ON DISSIPATIVE OPERATORS WITH ABSOLUTELY CONTINUOUS SPECTRUM

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Abstract

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ON DISSIPATIVE OPERATORS WITH ABSOLUTELY CONTINUOUS SPECTRUM

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§ 1. In the present paper we consider a non-self-adjoint operator A of class $i\Omega$ (^{1}) with continuous spectrum. In addition, we shall assume that the operator A is dissipative, i.e. $(A-A^*)/i \geq 0$. As M. S. Livshits (^{1}) showed, the characteristic matrix-function of such an operator is represented in the form

$$w(\lambda) = \int_0^l \exp\left[-\frac{i\beta^2(t)}{\alpha(t)-\lambda}\,dt\right],$$

where the function $\alpha(t)$ increases monotonically and is bounded, while the matrix $\beta(x)$ is nonnegative and $\operatorname{sp}\beta^2(x) \equiv 1$.

Assume additionally that the operator A has absolutely continuous spectrum, i.e. the function $t = \alpha(x)$ has an absolutely continuous inverse function $x = \sigma(t)$. In this case

$$w(\lambda) = \int_0^b \exp\left[-\frac{i\beta_1^2(t)}{t-\lambda}\,dt\right],$$

where $\beta_1(t) = p(t)\beta(\sigma(t))$, $p(t) = \sqrt{\sigma'(t)}$, $a = \alpha(0)$, $b = \alpha(l)$.

The triangular model of the operator A, as follows from (^{1}), can be written in the form

$$\vec{A}f = xf(x) + i\int_a^x f(t)\beta_1(t)\,dt\,\beta_1(x) \qquad (a \leq x \leq b). \tag{1}$$

Theorem 1. *The additional component of the operator $\vec{A}$ consists of those and only those vector-functions $f(x) \in L_r^2[a,b]$ for which, almost everywhere, the equality*

$$f(x)\beta_1(x) \equiv 0, \qquad \text{if } \|\beta_1(x)\| \leq M.$$

§ 2. For what follows, the behavior of the multiplicative integral

$$w(b,\lambda)=\int_a^b \exp\left[-\frac{i\beta_1^2(t)}{t-\lambda}\,dt\right] \qquad \left(\int_a^b \|\beta_1^2(t)\|\,dt<\infty\right) \tag{2}$$

as $\tau\to 0$ ($\lambda=\sigma+i\tau$) is essential.

Theorem 2. *For almost all $\sigma\in[a,b]$ there exist limiting values*

$$w^{\pm}(\sigma)=\lim_{\tau\to\pm 0} w(b,\lambda)$$

and the formulas hold

$$w^{\pm}(\sigma)=\lim_{\varepsilon\to 0}\int_a^{\sigma-\varepsilon}\exp\left[-\frac{i\beta_1^2(t)}{t-\gamma}\,dt\right]\exp[\pm\pi\beta_1^2(\sigma)]\int_{\sigma+\varepsilon}^{b}\exp\left[-\frac{i\beta_1^2(t)}{t-\gamma}\,dt\right], \tag{3}$$

where the limits are understood in the sense of strong convergence.

The theorem was previously proved by us under the condition that $\operatorname{sp}\beta_1^2(t)$ is bounded (^{2,3}).

From formula (3) it follows that

$$w^{\pm}(\sigma)=R^{\pm 1}(\sigma)u(b,\sigma), \tag{4}$$

where

$$R^{\pm 1}(\sigma)=\lim_{\varepsilon\to+0}\int_a^{\sigma-\varepsilon}\exp\left[-\frac{i\beta_1^2(t)\,dt}{t-\sigma}\right]\exp[\pm\pi\beta_1^2(\sigma)]\left[\int_a^{\sigma-\varepsilon}\exp\left[-\frac{i\beta_1^2(t)}{t-\sigma}\,dt\right]\right]^{-1}=$$

$$=\exp[\pm\pi r^2(\sigma)], \tag{5}$$

$$U(b,\sigma)=\lim_{\varepsilon\to+0}\int_a^{\sigma-\varepsilon}\exp\left[-\frac{i\beta_1^2(t)}{t-\sigma}\,dt\right]\int_{\sigma+\varepsilon}^{b}\exp\left[-i\frac{\beta_1^2(t)}{t-\sigma}\,dt\right]. \tag{6}$$

Let H and H_1 denote the closures of the manifolds into which $L_r^2[a,b]$ is mapped upon multiplication of its elements respectively by $\beta_1(x)$ and $R(x)-R^{-1}(x)$.

By Theorem 1, the operator $\vec{A}$ induces its simple part on H.

In what follows we shall consider the operator $\vec{A}$ only on the space H.

In $(^{4-6})$ we constructed mutually inverse operators B and B^{-1}, defined on dense sets respectively in H_1 and H, by means of the formulas

$$B\varphi = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_a^x \varphi(\sigma)\sqrt{R(\sigma)-R^{-1}(\sigma)}\,U(x,\sigma)\,d\sigma\,\beta_1^{-1}(x), \tag{7}$$

$$B^{-1}f = \frac{1}{\sqrt{2\pi}}\left\{\int_a^x [f(\sigma)\beta_1^{-1}(\sigma)]' U^*(\sigma,x)\,d\sigma + f(a)\beta_1^{-1}(a)\right\}\sqrt{R(x)-R^{-1}(x)}. \tag{8}$$

Here

$$U(x,\sigma)=\lim_{\varepsilon\to+0}\int_a^{\sigma-\varepsilon}\exp\left[-\frac{i\beta_1^2(t)}{t-\sigma}\,dt\right]\int_{\sigma+\varepsilon}^b \exp\left[-\frac{i\beta_1^2(t)}{t-\sigma}\,dt\right].$$

Formulas (7)–(8) can be given meaning also in the case when the matrix $\beta_1(x)$ has no inverse on a set of positive measure $(^{4-6})$.

Item 3. The relation holds

$$\vec{A} = BQB^{-1}, \tag{9}$$

where Q is the operator of multiplication by the independent variable,

$$Qf = xf, \qquad f \in H_1.$$

We note that, in deriving relations (7)–(9), in works $(^{4-6})$ we assumed $\operatorname{sp}\beta_1^2(t)$ to be bounded. However, it is easy to dispense with this condition by using Theorem 2.

If $\vec{A}$ and Q are connected by a relation of the type (9), then they are called **linearly similar** $(^{4-5})$. If, moreover, B and B^{-1} are bounded, then the operators $\vec{A}$ and Q are called **linearly equivalent**.

Theorem 3. The relations

$$\left\|e^{-\frac{\pi}{2}r^2}B^{-1}\right\| = 1, \qquad \left\|Be^{-\frac{\pi}{2}r^2}\right\| = 1,$$

hold, where the operator $e^{-\frac{\pi}{2}r^2}$ is defined by the formula

$$e^{-\frac{\pi}{2}r^2}f = f(x)e^{-\frac{\pi}{2}r^2(x)}.$$

Theorem 4. In order that the operator $\vec{A}$ be linearly equivalent to a self-adjoint operator, it is necessary and sufficient that

$$\operatorname{vrai\,sup} \|\beta_1^2(x)\| = M < \infty. \tag{10}$$

Corollary. If condition (10) is satisfied, then the operator $\vec{A}$ is equivalent to Q, and

$$\|B\| = e^{\frac{\pi}{2}M}, \qquad \|B^{-1}\| = e^{\frac{\pi}{2}M}.$$

In terms of the characteristic matrix-function, Theorem 4 admits the following reformulation.

Theorem 4′*. In order that a dissipative operator A of class $i\Omega$ with absolutely continuous spectrum be linearly equivalent to a self-adjoint operator, it is necessary and sufficient that

$$\operatorname{vrai\,sup} \|w^+(\sigma)\| < \infty.$$

Theorem 3 makes it possible to study the operator A also in the case when condition (10) is not satisfied.

Theorem 5. The multiplicity of the spectrum of the operator A is equal to

$$N = \operatorname{vrai\,sup} \operatorname{rang} \beta_1^2(x) = \operatorname{vrai\,sup} \operatorname{rang} [w^+(x) - w^-(x)].$$

Corollary. The operator A decomposes into the sum of N induced operators of first multiplicity.

Theorem 6. There exist invariant subspaces $H^{(n)}$ and $H_1^{(n)}$ of the operators A and Q, respectively, such that the operators $A^{(n)}$ and $Q^{(n)}$ induced on them are linearly equivalent. Moreover, the projection operators onto these subspaces satisfy

$$P^{(n)} \xrightarrow[n\to\infty]{} I, \qquad P_1^{(n)} \xrightarrow[n\to\infty]{} I.$$

All the results remain valid also for operators not belonging to the class $i\Omega$, whose characteristic matrix-function admits the representation (2).

Let us note that the theorems formulated here can be carried over to the case when one or both endpoints of the segment $[a, b]$ are infinite. In this case the operators A and Q are unbounded.

4. Let us study the behavior of e^{iAt} as $t \to \pm\infty$. We shall compare it with the behavior of $e^{iA_1 t}$ as $t \to \pm\infty$, where A_1 is the real component of A ($A_1 = (A + A^*)/2$). Let G be the subspace corresponding to the absolutely continuous part of the spectrum of A_1, and let P_G be the projection operator onto G.

* *Note added in proof.* A more general result has been obtained by B. S. Naim and K. Hoffman (^{10}).

Theorem 7. Let the prime operator A satisfy the conditions of Theorem 4. Then the strong limits exist

$$W_{\pm}(A, A_1) = \lim_{t\to\pm\infty} e^{iAt}e^{-iA_1t}P_G,$$

$$W_{\pm}(A_1, A) = \lim_{t\to\pm\infty} e^{iA_1t}e^{-iAt}.$$

Moreover, the operators $W_{\pm}(A, A_1)$ and $W_{\pm}(A_1, A)$ are bounded together with their inverses, and the relations hold

$$A = W_{\pm}(A, A_1)A_1W_{\pm}(A, A_1)^{-1}, \qquad A_1 = W_{\pm}(A_1, A)AW_{\pm}(A_1, A)^{-1},$$

$$W_{\pm}(A, A_1) = W_{\pm}^{-1}(A_1, A),$$

where A_1 is considered only on G.

Thus, Theorem 7 extends to non-self-adjoint operators the well-known Rosenblum-Kato theorem (^{7,8}). We note that some results in this direction were obtained earlier (^{9}).

Introduce the scattering operator S by the formula

$$S = W_{-}^{-1}(A, A_1)W_{+}(A, A_1).$$

The operator S is defined in G and commutes in G with A, $\|S\| \leqslant 1$.

Theorem 8. The operator S is unitarily equivalent to multiplication by the matrix

$$S(x) = I - \sqrt{R(x) - R^{-1}(x)}\,R^{1/2}(x)U(b, x)(I + R(x)U(b, x))^{-1}\times$$

$$\times R^{-1/2}(x)\sqrt{R(x) - R^{-1}(x)}$$

in the space H_1. Here R and U are related to the characteristic function w of the operator A by relations (4)–(6).

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Note: Figure translations are in progress. See original paper for figures.

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