ON THE REGULARIZED TRACE OF THE DIFFERENCE OF TWO SINGULAR STURM-LIOUVILLE OPERATORS

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ON THE REGULARIZED TRACE OF THE DIFFERENCE OF TWO SINGULAR STURM-LIOUVILLE OPERATORS

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As is known, I. M. Gel' fand and B. M. Levitan ([1]) were the first to obtain a formula for the sum of the differences of the eigenvalues of two regular Sturm–Liouville operators. Subsequently, in ([2–4]), analogues of these formulas were obtained in the case of certain singular Sturm–Liouville operators. In the present note these questions are investigated in connection with the works of M. G. Krein ([5,6]) from a unified point of view.

1. In the space $\mathcal{L}_2(0,\infty)$ we consider the operators H_0 and H_1 generated by the equalities

$$H_0 y = -y'' + q_0(x)y, \tag{1}$$

$$H_1 y = -y'' + q_0(x)y + q(x)y = -y'' + q_1(x)y \tag{2}$$

and by the boundary condition

$$y'(0) - hy(0) = 0. \tag{3}$$

It is assumed that $q_0(x)$ is a real locally integrable function such that H_0 is a self-adjoint operator; $q(x)$ is a real finite integrable function; h is a real number. For $h = \infty$, condition (3) is understood as $y(0) = 0$. It is easy to show that the difference of the resolvents of the operators H_0 and H_1 is a nuclear operator.

Consequently, from the theorem of M. G. Krein ([6]) it follows that there exists a real function $\xi(t) = \xi(t, H_0, H_1)$ such that, for any non-real z,

$$\mathrm{Sp}\{R_z(H_1) - R_z(H_0)\} = -\int_{-\infty}^{\infty} \frac{\xi(t)}{(t-z)^2}\, dt, \tag{4}$$

where $R_z(H) = (H - zI)^{-1}$.

From equality (4) the function $\xi(t)$ is determined almost everywhere up to a constant term.

Theorem 1. *The function $\xi(t)$ in equality (4) can be chosen so that, for any $a \geq 0$,*

$$\int_{-\infty}^{0} e^{a\sqrt{|t|}} |\xi(t)|\, dt < +\infty. \tag{5}$$

Moreover, if the Fourier cosine transform of the finite function $q(x)$ converges to $q(0)$ at the point $x = 0$, then

$$\int_{-\infty}^{0} \xi(t)\, dt + \int_{0}^{\infty} \left(\xi(t) - \frac{1}{2\pi\sqrt{t}} \int_{0}^{\infty} q(x)\, dx \right) dt = \frac{1}{4} q(0) \quad \text{for } h \neq \infty, \tag{6}$$

$$\int_{-\infty}^{0} \xi(t)\, dt + \int_{0}^{\infty} \left(\xi(t) - \frac{1}{2\pi\sqrt{t}} \int_{0}^{\infty} q(x)\, dx \right) dt = -\frac{1}{4} q(0) \quad \text{for } h = \infty. \tag{6'}$$

In ([7]) this theorem was proved under the condition that H_0 is a semibounded operator (in this case inequality (5) is trivial).

For the proof of Theorem 1 the following inequality is used, valid in the case $q(x) \geq 0$:

$$\int_{0}^{\infty} q(x)\theta_1(x, x, \lambda)\, dx \leq \int_{-\infty}^{\lambda} \xi(t)\, dt \leq \int_{0}^{\infty} q(x)\theta_0(x, x, \lambda)\, dx, \tag{7}$$

where $\theta_0(x, s, \lambda)$ and $\theta_1(x, s, \lambda)$ are the spectral kernels (see (8)) of the operators H_0 and H_1.

Inequality (5) is obtained if inequality (7) is multiplied by $ae^{a\sqrt{|\lambda|}}/2\sqrt{|\lambda|}$, where $a > 0$, and integrated with respect to λ from $-\infty$ to 0. In doing so one must use the known estimate (9) for the spectral function.

From the asymptotics for the spectral kernels $\theta_0(x, s, \lambda)$ and $\theta_1(x, s, \lambda)$ (see (8)) and from (13) it follows that, as $\lambda \to +\infty$,

$$\int_{-\infty}^{\lambda} \xi(t)\, dt - \frac{1}{\pi}\sqrt{\lambda} \int_{0}^{\infty} q(x)\, dx = \frac{1}{2\pi} \int_{0}^{\infty} q(x) \frac{\sin 2\sqrt{\lambda}x}{x}\, dx + o(1) \quad \text{for } h \neq \infty, \tag{8}$$

$$\int_{-\infty}^{\lambda} \xi(t)\, dt - \frac{1}{\pi}\sqrt{\lambda}\int_0^{\infty} q(x)\, dx = -\frac{1}{2\pi}\int_0^{\infty} q(x)\frac{\sin 2\sqrt{\lambda}x}{x}\, dx + o(1) \quad \text{for } h = \infty. \tag{8'}$$

These relations were obtained under the assumption $q(x) \geq 0$. However, this assumption is easily removed. Theorem 1 follows from (8) and (8′) and from the properties of Dirichlet integrals. In the case where the operator H_0 has a discrete spectrum bounded below, this theorem coincides with the result of M. G. Gasymov ([3]).

2. Consider the special case where $q_0(x) \equiv 0$ and $h = \infty$, requiring, instead of the finiteness of the function $q(x)$, only that $(1+x)q(x) \in \mathcal{L}_1(0,\infty)$.

Theorem 2. If $(1+x)q(x) \in \mathcal{L}_1(0,\infty)$ and the Fourier cosine transform of the function $q(x)$ at the point $x = 0$ converges to $q(0)$, then relation (6′) holds.

In the proof of this theorem the following lemma is used.

Lemma. If $(1+x)q(x) \in \mathcal{L}_1(0,\infty)$, then there exists a constant C such that for all $0 \leq x < +\infty$ and $\lambda < +\infty$ the inequality

$$\left|\theta_1(x,x,\lambda) - \frac{1}{\pi}\int_0^{\lambda}\frac{\sin^2\sqrt{t}\,x}{\sqrt{t}}\, dt\right| \leq C$$

is valid.

As is known, under the condition $q(x) \in \mathcal{L}_1(0,\infty)$ the asymptotic formula

$$\psi_1(x,\lambda) = \frac{A(k)}{k}\sin(kx - \eta(k)) + o(1), \qquad k^2 = \lambda, \quad x \to +\infty$$

holds. The equality (see ([10,11])) is valid

$$\xi(\lambda) = \begin{cases} \frac{1}{\pi}\eta\left(\sqrt{\lambda}\right), & \lambda > 0, \\ -\int_{-\infty}^{\lambda}\delta(t-\lambda_k)\, dt, & \lambda < 0, \end{cases}$$

where $\{\lambda_k\}$ is the sequence of all negative eigenvalues of the operator H_1, and $\delta(t)$ is the Dirac function.

This relation, together with Theorem 2, gives

Theorem 3. Under the assumptions of Theorem 2, the relation

$$\sum_k \lambda_k + \frac{2}{\pi}\int_0^{\infty}\lambda\left(\eta(\lambda) - \frac{1}{2\lambda}\int_0^{\infty} q(x)\, dx\right) d\lambda = -\frac{1}{4}q(0). \tag{9}$$

Formula (9), for a twice continuously differentiable function $q(x)$ and under the condition $(1+x^2)q(x) \in \mathcal{L}_1(0,\infty)$, was first rigorously proved

by L. D. Faddeev (see (2), and also (10, 12, 13)). Let us note that in the formulation of Theorem 1 of (2) there is an inaccuracy, namely, it is asserted there that formula (9) holds whenever $q(x)$ is continuous at the point $x=0$ and $xq(x) \in \mathcal{L}_1(0,\infty)$. The incorrectness of this assertion follows from the fact that, as (8′) shows, for equality (9) to hold under the condition $(1+x)q(x) \in \mathcal{L}_1(0,\infty)$, it is necessary and sufficient that the Fourier cosine transform of the function $q(x)$ converge at the point $x=0$ to $q(0)$.

3. In this section we shall consider two singular Sturm—Liouville operators which are defined by one and the same differential expression (1), but by different conditions at zero. In the case when these operators have a discrete spectrum, M. G. Gasymov and B. M. Levitan (4) obtained a formula for the sum of the differences of the eigenvalues of these operators. Here this formula is generalized (Theorem 5) to arbitrary Sturm—Liouville operators (not necessarily with discrete spectrum). For semibounded operators this generalization was obtained by the author in another way in (7). Here we also establish formulas which connect the spectral shift function $\xi(t, H_{h_1}, H_{h_2})$ and the spectral function.

Let the operator H_h be defined by equalities (1) and (3). Since the resolvents of the operators H_{h_1} and H_{h_2} differ by a one-dimensional operator, formula (4) is valid for these operators.

From equality (4) and $(1+|t|)^{-1}\xi(t, H_{h_1}, H_{h_2}) \in \mathcal{L}_1(0,\infty)$, the function $\xi(t, h_1, h_2) = \xi(t, H_{h_1}, H_{h_2})$ is determined uniquely. Denote by $\sigma_h(\lambda)$ the spectral function of the operator H_h, normalized by the condition $\sigma_h(-\infty)=0$. The following is valid.

Theorem 4. *The following formulas hold* ∗:

$$\int_{-\infty}^{\infty} \frac{\xi(t,h_1,h_2)}{t-\lambda}\,dt = \ln\left(1 + (h_2-h_1)\int_{-\infty}^{\infty} \frac{d\sigma_{h_1}(t)}{t-\lambda}\right), \tag{10}$$

$$\int_{-\infty}^{\lambda} \xi(t,h_1,h_2)\,dt = \int_{h_1}^{h_2} \sigma_h(\lambda)\,dh. \tag{11}$$

From the asymptotic formula for the spectral function (see (8, 14)) and from (11) there follows the following

Theorem 5. *The formula holds*

$$\int_{-\infty}^{0} \xi(t,h_1,h_2)\,dt + \int_{0}^{\infty}\left(\xi(t,h_1,h_2) - \frac{1}{\pi\sqrt{t}}(h_2-h_1)\right)dt = -\frac{h_2^2-h_1^2}{2}.$$

For every $a \geq 0$

$$\int_{-\infty}^{0} e^{a\sqrt{|t|}} |\xi(t)| \, dt < +\infty.$$

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* Formula (10) was communicated to the author by M. G. Krein.

Note: Figure translations are in progress. See original paper for figures.