ON PROJECTION SPECTRA AND BICOMPACT EXTENSIONS

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ON PROJECTION SPECTRA AND BICOMPACT EXTENSIONS

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The notion of a projection spectrum as the inverse spectrum of a countable set of finite complete simplicial complexes with simplicial mappings as projections was introduced by P. S. Aleksandrov ([1]); this notion was freed by A. G. Kurosh ([2]) from the requirement, entering into it, that the set of complexes be countable; finally, the notion of a projection spectrum received full generality in the works of V. I. Ponomarev ([3], [4]) (an arbitrary directed set of simplicial complexes of arbitrary cardinality). The Aleksandrov–Kurosh theorem states:

Every bicompactum (bicompact Hausdorff space) is the limit of some projection spectrum; conversely, the limit of any finite (i.e. consisting of finite complexes) spectrum is a bicompact T_1-space, possibly non-Hausdorff.

V. I. Ponomarev proved that every paracompactum (paracompact Hausdorff space) is the limit of some spectrum (but no longer a finite one). In fact these authors did more: they showed that every space considered by them (a bicompactum in Kurosh, a paracompactum in Ponomarev) is the limit of a certain quite definite spectrum, namely the maximal (respectively, maximal finite) spectrum of the given space. Under this name V. I. Ponomarev was the first to define with complete clarity the spectrum whose complexes are the nerves of all, respectively all finite, coverings of the given space. Here by a covering is meant any locally finite covering α, whose elements are canonical closed sets $A_\lambda^\alpha = [U_\lambda^\alpha]$ with disjoint open kernels $U_\lambda^\alpha = IA_\lambda^\alpha$. The order and the projections are defined naturally: $\alpha' > \alpha$, if every element $A_\lambda^{\alpha'}$ of the covering α' is contained in some (obviously unique) element A_λ^α of the covering α; by associating with the element $A_\lambda^{\alpha'}$ the element A_λ^α of the covering α that contains it, we obtain a simplicial mapping (projection) of the nerve α' onto the nerve α. For bicompact spaces the notions of maximal spectrum and maximal finite spectrum coincide, since a locally finite covering of a bicompact space is always finite. Thus every bicompactum (Kurosh) and every paracompactum (Ponomarev) is the limit of its maximal projection spectrum. On the other hand, the limit of every projection spectrum is a T_1-space, bicompact in the case of finite spectra; however, not every bicompact T_1-space is the limit of its maximal spectrum.

In this paper, by topological spaces we shall always mean T_1-spaces. Recall that a T_1-space is called semiregular if the canonical open sets form a base in it (or, equivalently, the canonical closed sets form a closed base). For brevity, we shall call semiregular T_1-spaces T_ξ-spaces. T_ξ-spaces need not be Hausdorff; at the same time not every Hausdorff space is a T_ξ-space.

Finally, the following two results substantially complete the whole picture:

The limit of the maximal finite spectrum of a normal space X is the Čech extension βX (P. S. Aleksandrov).

The limit of the maximal finite spectrum of any regular space X is the bicompact T_1-extension $\omega_\chi X$ of Wallman type introduced by Ponomarev ([5]): the points of the space $\omega_\chi X$ are χ-systems, i.e. maximal centered systems of canonical closed sets of the space X; the topology in $\omega_\chi X$ is the usual Wallman topology. We shall return to this last result below.

In the present note the following facts are established.

Theorem 1. *If the maximal finite spectrum S_X of a given T_1-space X exists (i.e. if the finite decompositions of the space X, with their natural order, form a directed set), then the limiting space $\widetilde{S}_X$ of the spectrum S_X is naturally homeomorphic to the space $\omega_\chi X$.*

The proof of this theorem is based on the following considerations. Along with the Wallman–Ponomarev space we define the space $\omega_\pi X$, whose points are π-systems, i.e. maximal centered systems of sets P, each of which is the intersection of a finite number of canonical closed sets (the topology is the same).

First of all one proves

Lemma 1. *The spaces $\omega_\pi X$ and $\omega_\chi X$ are naturally homeomorphic to each other.*

For the proof of Lemma 1 we take an arbitrary π-system ξ and, in it, the subsystem $\chi\xi$ consisting of canonical closed sets alone.

It turns out that ξ consists of all finite intersections of elements of the system $\chi\xi$, and that $\chi\xi$ is a χ-system.

Assigning to each π-system the unique χ-system contained in it, we obtain a natural homeomorphism between the spaces $\omega_\pi X$ and $\omega_\chi X$.

V. I. Ponomarev, from analogous considerations, even derives a homeomorphism between his space $\omega_\chi X$ and the original Wallman space ωX. However, his reasoning on this point does not seem convincing to me: the spaces ωX and $\omega_\chi X$, generally speaking, are distinct.

For the proof of Theorem 1 one can, relying on Lemma 1, prove that the space $\widetilde{S}_X$ is homeomorphic to the space $\omega_\pi X$, and apply Ponomarev' s method to this in the appropriate way.

Let $\xi = \{P_\lambda\}$ be a point of the space $\omega_\pi X$. In each decomposition $\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$ of the space X there is an element A_i^α which intersects all $P_\lambda \in \xi$, and then $A_i^\alpha \in \xi$. Let $A_{i_0}^\alpha, \dots, A_{i_r}^\alpha$ be all the elements of the covering α each of which intersects all $P_\lambda \in \xi$. Then $A_{i_0}^\alpha, \dots, A_{i_r}^\alpha$ are contained in ξ, and, consequently,

$$A_{i_0}^\alpha \cap \dots \cap A_{i_r}^\alpha \neq \Lambda$$

–the elements $A_{i_0}^\alpha, \dots, A_{i_r}^\alpha$ form a vertex of the complex $t_\alpha = t_\alpha(\xi)$ of the nerve of α. Performing this construction for each α, we obtain a thread $\eta(\xi) = \{t_\alpha(\xi)\}$ of the spectrum S_X, and this thread proves to be maximal. Thus, to each point $\xi \in \omega_\pi X$ there corresponds a point $\eta = \eta(\xi) \in \widetilde{S}_X$, and the resulting mapping of the space $\omega_\pi X$ into the space $\widetilde{S}_X$ is, as can be shown, the required homeomorphism between $\omega_\pi X$ and $\widetilde{S}_X$.

Since for every T_ξ-space X the maximal finite spectrum exists, Theorem 1 implies

Corollary. *The limit of the maximal finite spectrum of every T_ξ-space X is the space $\omega_\chi X$.*

We now pass to the question: when is a given bicompact space X the limit of its maximal finite spectrum?

It is proved first of all:

Theorem 2a. *If X is a bicompact T_ζ-space, then the natural mapping*

$$\varphi: \ X \to \check{S}_X$$

is a homeomorphism.

Here by the natural mapping $\varphi: \ X \to \check{S}_X$ is meant the mapping constructed as follows: for each point $x \in X$, in each partition

$$\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$$

we take the set of all elements of this partition containing the point x. We obtain a simplex $t_\alpha(x)$ of the nerve $|\alpha|$. The simplexes $t_\alpha(x)$, constructed for all α, form a maximal thread of the spectrum S_X of the space X, i.e. a point of the space $\check{S}_X$, which is thus put into correspondence with the point x. Next, we have

Theorem 2b. *If the space X is naturally homeomorphic to the limit of its maximal finite spectrum $\check{S}_X$, then it is a T_ζ-space.*

Combining Theorems 2a and 2b, we obtain the following result:

Theorem 2. *A bicompact space X is naturally homeomorphic to the limit of its maximal spectrum S_X if and only if X is a T_ζ-space.*

Obviously, every regular space is a T_ζ-space in which the canonical closed sets form a net.*

Let us call a T_λ**-space** any T_ζ-space in which the canonical closed sets form a pseudonet.*

For what follows we shall need

Lemma 2. *Let X be a T_λ-space, $\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$ its finite partition. For each closed canonical set A in X, denote by $\bar{A}$ the closure of A in $\omega_\varkappa X$.*

Then, if $\bar{A}_{i_1}^\alpha \cap \dots \cap \bar{A}_{i_r}^\alpha \neq \Lambda$, then also $A_{i_1}^\alpha \cap \dots \cap A_{i_r}^\alpha \neq \Lambda$.

Indeed, if $\xi = \{A_\lambda\} \in \bar{A}_{i_1}^\alpha \cap \dots \cap \bar{A}_{i_r}^\alpha$, then $A_k^\alpha \in \xi$ for $k = i_1, \dots, i_r$; hence

$$A_{i_1}^\alpha \cap \dots \cap A_{i_r}^\alpha \neq \Lambda.$$

From this simple lemma it follows that the nerves of the coverings $\alpha = \{A_1^\alpha, \dots, \dots, A_{s_\alpha}^\alpha\}$ and $\bar{\alpha} = \{\bar{A}_1^\alpha, \dots, \bar{A}_{s_\alpha}^\alpha\}$, respectively, of the spaces X and $\omega_\varkappa X$, are isomorphic, and then the maximal finite spectra of the spaces X and $\omega_\varkappa X$ are also isomorphic. Hence and from Theorem 1 we obtain

Theorem 3. *For any T_λ-space X, the space $\omega_\varkappa X$ is homeomorphic to the limit of its maximal finite spectrum. It turns out that this homeomorphism is natural. Therefore, relying on Theorem 2, we conclude that $\omega_\varkappa X$ is a T_ζ-space.*

It can also be shown that $\omega_\varkappa X$ is an extension of the T_λ-space X. Thus, we have:

Theorem 4. *For any T_λ-space X (in particular, for any regular space), the space $\omega_\varkappa X$ is a bicompact T_ζ-extension.*

It follows from Theorem 4 that every T_λ-space has a bicompact–

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* A net of a space X in the sense of A. V. Arhangel' skii is any system $\mathfrak{M}$ of sets $M \subseteq X$ satisfying the condition: whatever the point x and its neighborhood Ox, there exists an $M \in \mathfrak{M}$ such that $x \in M \subseteq Ox$. A system $\mathfrak{M}$ of sets $M \subseteq X$ is called a pseudonet if, by adjoining to it all possible finite intersections $M_1 \cap \dots \cap M_s$ of its elements, we obtain a net.

a T_ξ-extension. The converse assertion is also true. In other words:

Theorem 5. *In order that a space X have a bicompact T_ξ-extension, it is necessary and sufficient that X be a T_λ-space.*

The necessity of the condition contained in this theorem follows from the following two lemmas.

Lemma 3. *The canonical closed sets in a bicompact T_ξ-space form a pseudonet.*

Lemma 4. *If the canonical closed sets form a pseudonet in a bicompact extension $\overline{X}$ of a space X, then the canonical closed sets of the space X also form a pseudonet.*

Consider some T_λ-space X and its bicompact T_ξ-extension ξX. In view of the fact that the closure operator in the space ξX establishes a one-to-one correspondence between the canonical closed sets of the spaces X and ξX, and also establishes a one-to-one correspondence between the finite partitions α and $\overline{\alpha}$ of these spaces, the nerve of the partition $\overline{\alpha}$ is an "extension" of the nerve α (the two nerves have one and the same set of vertices, but $\overline{\alpha}$ may have additional simplexes in comparison with α). Hence the spectrum of the space ξX is also an extension of the spectrum of the space X and of the spectrum of the space $\omega_\lambda X$ isomorphic to it.

Each point of the space $\omega_\lambda X$ may be regarded as a maximal thread ξ^x of the spectrum $S_X = S_{\omega_\lambda X}$; this thread is a thread ξ^x also of the spectrum $S_{\xi X}$ of the space ξX, generally speaking, not maximal. In the case when the thread is not a maximal thread, we take the unique maximal thread containing it. This establishes a natural mapping $\xi : \omega_\lambda X \to \xi X$ of the space $\omega_\lambda X$ into ξX, which we call the **spectral mapping**. Under this mapping every $\overline{A}_i^\alpha = [A_i^\alpha]_{\omega_\lambda X}$ is mapped into $[A_i^\alpha]_{\xi X} = \xi \overline{A}_i^\alpha$.

It follows easily from this that the spectral mapping

$$\xi : \ \omega_\lambda X \to \xi X$$

is θ-continuous. Further, it is a mapping of the space $\omega_\lambda X$ onto the whole space ξX; finally, all points of X remain fixed under the mapping ξ.

Thus we have:

Theorem 6. *Among all bicompact T_ξ-extensions of a given regular space X, the Wallman–Ponomarev extension $\omega_\lambda X$ is the unique maximal one in the sense that the spectrum of the space ξX is an extension of the spectrum of the space $\omega_\lambda X$; there exists a natural mapping of the space $\omega_\lambda X$ onto ξX. This mapping is θ-continuous and leaves all points of X fixed.*

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Note: Figure translations are in progress. See original paper for figures.