SUMMATION OF UNBOUNDED SEQUENCES BY LINEAR REGULAR METHODS

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Abstract

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SUMMATION OF UNBOUNDED SEQUENCES BY LINEAR REGULAR METHODS

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1°. The summation, by regular matrices, of unbounded sequences is studied. The methods of investigation used are, in particular, related to the methods of ([1], [2]). Since a linear regular summability method is determined by a regular matrix, we often say "matrix" instead of "method," meaning by this the method determined by this matrix, and conversely.

2°. The set of bounded sequences summable by a regular matrix A forms its field $F(A)$; the set of all sequences summable by a regular matrix A forms its domain of effectiveness $G(A)$. A linear regular summability method B is boundedly stronger than a linear regular summability method A (the matrix B is boundedly stronger than the matrix A) if $F(B) \supset F(A)$; a linear regular summability method B is stronger than a linear regular summability method A (the matrix B is stronger than the matrix A) if $G(B) \supset G(A)$. Unbounded (respectively bounded) sequences $\{s^1\}, \{s^2\}, \ldots, \{s^n\}$ are distinct (linearly independent) if every nontrivial linear combination of these sequences is an unbounded (respectively divergent) sequence. Unbounded (respectively bounded) sequences $\{s^1\}, \{s^2\}, \ldots, \{s^n\}$ are linearly independent relative to the matrix A if every nontrivial linear combination of the A-transforms of these sequences is an unbounded (respectively divergent) sequence.

3°. In the well-known Mazur-Orlicz theorem from ([3]), theorem 3 (for the proof see ([4], [5])), it is shown that if a regular matrix A sums a bounded sequence, then it also sums an unbounded sequence; i.e., if $F(A) \supset F(E)$ (E is the identity matrix), then the set $\{G(A) - F(A)\}$, in any case, is nonempty. But how large is the set of distinct sequences forming $\{G(A) - F(A)\}$? From Theorem 1 it follows that this set has the cardinality of the continuum.

Theorem 1. *If a regular matrix A sums at least one bounded divergent sequence, then one can construct a continuum set of sequences, unbounded together with any finite nontrivial linear combination of them, such that each of the sequences belonging to this set is summed by the matrix A.*

Let now a regular matrix B sum at least one bounded divergent sequence not summable by the regular matrix A. What can be asserted in this case about the set of divergent sequences summable by the matrix B?

Theorem 2. *If a regular matrix B sums at least one bounded divergent sequence not summable by the regular matrix A, then one can construct a continuum set of sequences whose A-transforms are unbounded together with any finite nontrivial linear combination of them, such that each*

a sequence belonging to this set is summable by the matrix B.

If, in particular, the regular matrix B is boundedly stronger than the regular matrix A (i.e., $F(B) \supset F(A)$), then from this and from Theorem 3 of [1] (see also [2]) it follows that

Theorem 3. *If a regular matrix B is boundedly stronger than a regular matrix A, then the set $\{G(B)\}$ contains effectively constructible: 1) a continuum set of bounded sequences that are linearly independent relative to the matrix A; 2) a continuum set of unbounded sequences that are linearly independent relative to the matrix A.*

4°. The following remark supplements Theorem 3. In this theorem it is proved that from $F(B) \supset F(A)$ it follows that $G(B)$ differs from $G(A)$ by a continuum set of unbounded sequences that are linearly independent relative to the matrix A. It is interesting that in the case $F(B) \equiv F(A)$, the field of effectiveness $G(B)$ may differ from $G(A)$ by only a finite number of unbounded sequences that are linearly independent relative to the matrix A. We give the corresponding example.

Denote by R the regular matrix corresponding to the linear transformation

$$\sigma_n = \sum_{k=0}^{p} \alpha_k t_{n-(p-k)}, \qquad \alpha_k = \binom{p}{k} (-1)^k 2^{p-k}, \quad k = 0, 1, 2, \dots, p,$$

of the sequence $\{t_n\}$ into the sequence $\{\sigma_n\}$. Let, further, $D = \{d_{n,k}\}$ be a normal regular matrix that does not sum any unbounded sequence of the form $\{n^k \cdot 2^n\}$, $k = 0, 1, 2, \dots, p-1$ (for example, $d_{n,k} = 1/(n+1)$, $k \le n$; $d_{n,k} = 0$, $k > n$). Put $t_n = \sum_{k=0}^{n} d_{n,k} s_k$. Form the regular matrix $Q = R \cdot D$, corresponding to the transformation of the sequence $\{s_n\}$ into the sequence $\{\sigma_n\}$. The field of the matrix Q is identical with the field of the matrix D, i.e. $F(Q) = F(D)$, while its field of effectiveness $G(Q)$ consists of the field $G(D)$ and the unbounded sequences $\{s_n^k\}$, $k = 0, 1, 2, \dots, p-1$, defined from the relation $\{n^k \cdot 2^n\} = D\{s_n^k\}$, $k = 0, 1, \dots, p-1$, linearly independent relative to the matrix D.

5°. From Theorem 3 of [1] (see also [2]) it follows that there is no regular matrix whose field of effectiveness is formed from the linear hull of a given finite number of linearly independent bounded divergent sequences and the trivial field $F(E)$. The situation will be essentially different if unbounded sequences are considered. Namely, as follows from Theorem 4, whatever finite number of unbounded linearly independent sequences there may be, one can always construct a regular matrix A whose field of effectiveness consists of the linear hull of these sequences and the trivial field $F(E)$.

Theorem 4. *Whatever p unbounded linearly independent sequences $\{s^1\}, \{s^2\}, \dots, \{s^p\}$ and numbers $x_1, x_2, \dots, x_p$ may be, it is possible to effectively construct a regular matrix A such that: 1) the matrix A sums the sequence $\{s^k\}$ to the number x_k, where $k = 1, 2, \dots, p$; 2) if the sequence of A-transforms of a sequence $\{x\}$ converges (i.e., $\{x\}$ is A-summable), then the sequence $\{x\}$ necessarily has the form*

$$\{x\} = \sum_{k=1}^{p} c_k \{s^k\} + \{s^*\},$$

where c_k, $k = 1, 2, \dots, p$, are certain constants and $\{s^*\}$ is some convergent

sequence; 3) if the sequence of A-transforms of the sequence $\{x\}$ is bounded, then the sequence $\{x\}$ necessarily has the form

$$\{x\} = \sum_{k=1}^{p} c'_k \{s^k\} + \{s^{**}\},$$

where c'_k, $k = 1, 2, \dots, p$, are certain constants and $\{s^{**}\}$ is a certain bounded sequence.

In the limiting case $p = 1$, assertions 1) and 2) of Theorem 4 were established in [5].

6°. From Theorem 1 there follows the following negative result, supplementing the preceding theorem.

Theorem 5. *If arbitrary unbounded sequences $\{s^1\}, \{s^2\}, \dots, \{s^p\}$ are such that there exists at least one nontrivial linear combination of these sequences which forms a bounded divergent sequence, then there is no regular matrix A whose domain of effectiveness would consist of the linear span of these sequences and the trivial field $F(E)$.*

From Theorem 4 and Theorem 5 it follows that

Theorem 6. *Let a finite number p of unbounded sequences $\{s^1\}, \{s^2\}, \dots, \{s^p\}$ be given. In order that there exist a regular matrix A whose domain of effectiveness $G(A)$ would consist of the linear span of these sequences and the trivial field $F(E)$, it is necessary and sufficient that these sequences be linearly independent.*

7°. The theorems of the present section are connected with the summation of sequences that are linearly independent with respect to a given regular matrix A.

On the basis of Theorem 4, one can establish the following theorem:

Theorem 7. *Whatever* $p+q$ *unbounded sequences* $\{s^1\}, \{s^2\}, \dots, \{s^p\}, \{s^{p+1}\}, \{s^{p+2}\}, \dots, \{s^{p+q}\}$ *linearly independent with respect to a regular matrix* A*, and whatever real numbers* $x_1, x_2, \dots, x_p$*, there exists an effectively constructible regular matrix* B *which sums the sequences* $\{s^1\}, \{s^2\}, \dots, \{s^p\}$ *to the limits equal to* $x_1, x_2, \dots, x_p$*, and which does not sum the sequences* $\{s^{p+1}\}, \{s^{p+2}\}, \dots, \{s^{p+q}\}$*.*

Combining this theorem with Theorem 11 from [6], one can establish the following theorem:

Theorem 8. *Whatever* $p+q$ *unbounded sequences* $\{s^1\}, \{s^2\}, \dots, \{s^p\}, \{s^{p+1}\}, \dots, \{s^{p+q}\}$ *linearly independent with respect to a regular matrix* A*,* p *real numbers* $x_1, x_2, \dots, \dots, x_p$*,* $(m+n)$ *bounded sequences* $\{\sigma^1\}, \{\sigma^2\}, \dots, \{\sigma^m\}, \{\sigma^{m+1}\}, \dots, \{\sigma^{m+n}\}$ *linearly independent with respect to the same regular matrix* A*, and* m *real numbers* $y_1, y_2, \dots, y_m$*, there exists a regular matrix* B *which: 1) sums the unbounded sequences* $\{s^1\}, \{s^2\}, \dots, \{s^p\}$ *respectively to the numbers* $x_1, x_2, \dots, x_p$ *and does not sum the unbounded sequences* $\{s^{p+1}\}, \{s^{p+2}\}, \dots, \{s^{p+q}\}$*; 2) sums the bounded sequences* $\{\sigma^1\}, \{\sigma^2\}, \dots, \{\sigma^m\}$ *respectively to the numbers* $y_1, y_2, \dots, y_m$ *and does not sum the bounded sequences* $\{\sigma^{m+1}\}, \{\sigma^{m+2}\}, \dots, \{\sigma^{m+n}\}$*.*

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Note: Figure translations are in progress. See original paper for figures.