



Soviet-era science, translated into English

ON HYPOELLIPTIC PSEUDODIFFERENTIAL OPERATORS

MATHEMATICS

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.94074>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.946

MATHEMATICS

Yu. V. EGOROV

ON HYPOELLIPTIC PSEUDODIFFERENTIAL OPERATORS

(Presented by Academician I. G. Petrovskii, 9 X 1965)

In the present note we generalize the results of L. Hörmander' s paper (1) on the hypoellipticity of differential operators with variable coefficients to the case of pseudodifferential operators (otherwise, singular integro-differential, or convolution operators). In doing so we use the technique developed in the work of J. Kohn and L. Nirenberg (2). Similar questions are considered in the note (3) of L. R. Volevich, but, unlike him, we do not assume that the operator under consideration has the same strength at different points.

1. We use the following notation: $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ are points of the n -dimensional real space R^n ; $(x, \xi) = x_1\xi_1 + \dots + x_n\xi_n$; $S(R^n)$ is the space of infinitely differentiable functions $f(x)$ decreasing at infinity faster than any power $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$. By $S'(R^n)$ we denote the space of linear functionals on $S(R^n)$, and by $\tilde{f}(\xi)$ the Fourier transform of $f(x)$ (see (4))

$$\tilde{f}(\xi) = \int f(x)e^{-i(x,\xi)} dx.$$

Finally, H^s , for any s , is the space of functions defined in R^n , with norm

$$\|u\|_s = \left(\int |\tilde{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.$$

Let Ω be a domain in R^n . A complex-valued function $p(x, \xi)$ ($x \in \Omega$, $\xi \in R^n$) is called the **symbol of the pseudodifferential operator**

$$Pu(x) = (2\pi)^{-n} \int p(x, \xi)\tilde{u}(\xi)e^{i(x,\xi)} d\xi.$$

Definition. A pseudodifferential operator P is called **hypoelliptic** in the domain Ω if from the condition $u(x) \in S'(R^n)$, $Pu(x) \in C^\infty(\Omega)$, it follows that $u(x) \in C^\infty(\Omega)$.

2. We shall assume that the operators under consideration satisfy the following Hörmander condition HE (see (1)).

Definition. The operator P satisfies condition **HE** in the domain Ω if:

- 1) $p(x, \xi) \in C^\infty(\Omega \times R^n)$.
- 2) In the domain $\Omega \times R^n$ there are defined functions $M_j(x, \xi)$ such that, for all α, β ,

$$|\partial^{\alpha+\beta} p(x, \xi) / \partial x^\beta \partial \xi^\alpha| \leq C_{\alpha, \beta, x} |\xi|^{m_x - d|\alpha|} M^{\beta' - \alpha}(x, \xi), \quad |\xi| \geq A_x,$$

$$1 \leq M_j(x, \xi) \leq C_x (1 + |\xi|)^{1-d}, \quad j = 1, \dots, n.$$

Here $d > 0$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$,

$$M^{\beta - \alpha} = M_{\beta_1} \dots M_{\beta_n} M_{\alpha_1}^{-1} \dots M_{\alpha_n}^{-1},$$

and the quantities C_x , $C_{\alpha, \beta, x}$, $n_{x, j}$, m_x , and A_x are bounded when x varies over compact subsets of Ω ; $\beta'_j = \min(\beta_j, n_{x, j})$.

- 3) For every compact subset $K \subset \Omega$ there are defined nonnegative constants $m(K)$ and $C(K)$ such that

$$|p(x, \xi)| \leq C(K) |\xi|^{m(K)}, \quad |p(x, \xi)|^{-1} \leq C(K) |\xi|^{m(K)}$$

for $x \in K$, if $\xi \in R^n$, $|\xi| \geq A_x$.

Remark. Condition 3), in particular, means that $p(x, \xi) \neq 0$ for $|\xi| \geq A_x$. In the case when P is a differential operator, i.e. $p(x, \xi)$ is a polynomial in ξ , condition 3) follows from 2), if only $p(x, \xi)$ does not vanish identically in any component of Ω (see (1)).

The main result of the present note is the following

Theorem. *If the operator P satisfies conditions 1)–3) in the domain Ω , then it is hypoelliptic in Ω .*

3. The proofs of the following Lemmas 1–3 are analogous to the proofs of the corresponding assertions in (2).

Lemma 1. *If the operators P and Q with symbols $p(x, \xi)$ and $q(x, \xi)$ satisfy conditions 1)–3), and $\zeta(x) \in C_0^\infty(\Omega)$, then the operator $Q\zeta(x)P$ is a pseudodifferential operator whose symbol in Ω is equal to*

$$s(x, \xi) = \sum_{|\alpha|=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha \zeta(x) p(x, \xi)}{\partial x^\alpha} \frac{\partial^\alpha q(x, \xi)}{\partial \xi^\alpha} + t_N(x, \xi)$$

and the order of the operator corresponding to the symbol $t_N(x, \xi)$ tends to $-\infty$ as $N \rightarrow \infty$.

Lemma 2. *If the operator P satisfies condition HE in Ω , $u(x), v(x) \in C_0^\infty(\Omega)$, the support of $v(x)$ lies in a fixed subdomain $\Omega_1 \subset \Omega$ with compact closure, and the support of $u(x)$ is at a positive distance from Ω_1 , then*

$$|(Pu, v)| \leq C_s \|u\|_s \|v\|_s$$

for every real s .

Lemma 3. *If $u \in S'(R^n)$, with $u \in C^\infty(\omega)$, where $\omega \subset \Omega$, and P satisfies condition HE in Ω , then $Pu \in C^\infty(\omega)$.*

Let us note that Lemma 3 implies the correctness of our definition of hypoellipticity.

4. Define in Ω a sequence of functions $q_0(x, \xi), \dots, q_j(x, \xi), \dots$, putting

$$q_0(x, \xi) = p(x, \xi)^{-1} \psi(\xi),$$

$$q_{j+1}(x, \xi) = -p(x, \xi)^{-1} \sum_{|\alpha|=1}^N \frac{1}{\alpha!} \frac{\partial^\alpha \zeta(x) p(x, \xi)}{\partial x^\alpha} \frac{\partial^\alpha q_j(x, \xi)}{\partial \xi^\alpha},$$

where $\psi(\xi) \in C^\infty(R^n)$, with $\psi(\xi) = 0$ for $|\xi| \leq A_x$ and $\psi(\xi) = 1$ for $|\xi| \geq A_x + 1$, and the number N_j is chosen so that the operator T , whose symbol is equal to the difference between the symbol of the operator $Q_j \zeta(x) P$ and

$$\sum_{|\alpha| \leq N_j} \frac{1}{\alpha!} \frac{\partial^\alpha p(x, \xi)}{\partial x^\alpha} \frac{\partial^\alpha q_j(x, \xi)}{\partial \xi^\alpha},$$

has negative order. Let ω be an arbitrary compact subdomain in Ω , and let $\zeta(x) \in C_0^\infty(\Omega)$ be a function equal to one in ω . Consider the operator Q , whose symbol is equal to $[q_0 + \dots + q_j] \zeta(x)$, and the operator R_j with symbol $[p(x, \xi) q_{j+1}(x, \xi) + 1 - \psi(\xi)]$. Then, evidently, every function $\varphi(x) \in C_0^\infty(\omega)$ is represented in the form

$$\varphi(x) = [P^* \zeta(x) Q^* + R_j^* + S^* + T^*] \varphi(x),$$

where the operator T^* has negative order, and the symbol of the operator S is equal to zero in ω .

Using condition HE and the construction of Q , it is not difficult to prove the following assertions (see ⁽¹⁾).

Lemma 4.

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \partial x^\beta} q_j(x, \xi) \right| \leq C_{\alpha, \beta, x, j} |\xi|^{m_x - d(|\alpha|+j)} M^{\beta-\alpha}(x, \xi),$$

$$\left| \frac{\partial^\beta}{\partial x^\beta} p(x, \xi) q_j(x, \xi) \right| \leq C_{\beta, x, j} |\xi|^{m_x + |\beta| - dj},$$

if $x \in \Omega$, $|\xi| \geq A_x$.

5. Proof of the theorem. Let $u \in S'(R^n)$, $Pu(x) \in C^\infty(\Omega)$. It is easy to see that $P(\zeta u) = g + h$, where $g \in C_0^\infty(\Omega)$, $h = 0$ in ω . Indeed, by Lemma 3, $P(u - \zeta u) \in C^\infty(\omega)$. Hence $g = P(\zeta u) \in C^\infty(\omega)$. Extend g from ω to a finite infinitely differentiable function in Ω . Then $h \in S'(R^n)$ and $h = 0$ in ω . Let j be so large that the order of the operator R_j is negative. For $\varphi \in C_0^\infty(\omega)$ we have

$$\begin{aligned} (\zeta u, \varphi) &= (\zeta u, (P^*Q^* + R_j^* + T^*)\varphi) = \\ &= (Q(g + h), \varphi) + ((R_j + T)\zeta u, \varphi). \end{aligned}$$

Consequently, $\zeta u = Q(g + h) + (R_j + T)\zeta u$. Since $h = 0$ in ω , $Qh \in C^\infty(\omega)$. Since $g \in C^\infty(\omega)$, also $Qg \in C^\infty(\omega)$. Since the order of $R_j + T$ is negative, $u \in C^\infty(\omega)$, as was required to prove.

Moscow State University
named after M. V. Lomonosov

Received
8 X 1965

REFERENCES

¹ L. Hörmander, Ann. Inst. Fourier, **11**, 477 (1961). ² J. J. Kohn, L. Nirenberg, Comm. Pure and Appl. Math., **18**, No. 1–2, 269 (1965). ³ L. R. Volevich, DAN, **168**, No. 6 (1966). ⁴ I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.