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Abstract

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ON THE GROUP CLASSIFICATION BY CARTAN' S METHOD OF THE EQUATIONS OF ONE-DIMENSIONAL GAS FLOW

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Suppose that we have some system of differential equations W with M functions u^l $(l, m, n = 1, 2, \dots, M)$ and $N - M$ independent variables x^i $(i, j, k = 1, 2, \dots, N - M)$. The space with coordinates x^i, u^l will be denoted briefly by $T(x, u)$, the differentials dx^i, du^l by dT, and all the variables x^i, u^l by y^α $(\alpha, \beta, \gamma = 1, 2, \dots, N)$. The study of the group properties of differential equations was begun by S. Lie and widely developed in the works of L. V. Ovsyannikov ([1−4]), and the methods developed by him were applied to the study of certain specific equations ([5−8]). However, the method of exterior forms of É. Cartan can also be applied to the study of the group properties of differential equations. In a very general formulation this was done by A. M. Vasil' ev ([9]), and in works ([10,11]) the group properties of a system of equations describing one-dimensional gas flows with plane waves were studied by É. Cartan' s method. As it turned out, this problem reduced to the construction of a transitive group for which the structural coefficients are constants.

In studying, by É. Cartan' s method, the group properties of differential equations in the presence of a finite group G_r admitted by the given system, two cases arise: a) G_r is a transitive group, b) G_r is an intransitive group. In both cases É. Cartan' s structural equations have the form:

$$D\omega^a = C^a_{bc}\omega^b \wedge \omega^c \qquad (a, b, c = 1, 2, \dots, r) \tag{1*}$$

(ω^a are the basic invariant forms of the group G_r, $\wedge$ is the symbol of exterior multiplication), but in case a) the coefficients $C^a_{bc} = \text{const}$, whereas in case b) they depend on ξ^α, where ξ^α denotes the coordinates in the representation space of the group G_r.

When studying concrete systems of equations, they may contain unfixed parameters Γ^τ and functions $F^\sigma(y)$ ([4]). The study of such systems by É. Cartan’ s method leads to the fact that the structural coefficients C^a_{bc} will also depend on Γ^τ, F^σ, and their derivatives up to some order. The character of the dependence of the coefficients C^a_{bc} on Γ^τ, F^σ, and their derivatives makes it possible to consider the classification problem using É. Cartan’ s method.

The construction of the group G_r admitted by the given system W is connected with the use of the canonization process ([12]), and canonization can be carried out in two ways. Indeed, suppose that we have some system W of differential equations

$$A^{mi}_l u^l_{x_i} + B^m = 0; \qquad A^{mi}_l = A^{mi}_l(x,u); \qquad B^m = B^m(x,u). \tag{2*}$$

Replacing system (2*) by a system of Pfaff forms ω^m and differentiating these forms exteriorly, we obtain equations

$$D\omega^m = b^m_{ln}\omega^l \wedge \omega^n + d^m_{lq}\omega^l \wedge \omega^q; \qquad q = 1, 2, \dots, q^1, \tag{3*}$$

where b^m_{ln}, d^m_{lq} depend on (x,u) and on the parametric derivatives $u^m_{x_i}$; q^1 is the number of parametric derivatives. Suppose there exists a linear transformation $\Omega^m = A^m_l\omega^l$ with a nonsingular matrix A^m_l such that it does not change the form of system (3*); in this case the coefficients b^m_{ln}, d^m_{lq} pass into B^m_{ln}, D^m_{lq}, which will depend on A^m_l. We choose these parameters A^m_l

so that as many coefficients B^m_{ln} as possible take constant values—this is the first method of canonization. In the second method of canonization one may choose the parametric derivatives themselves so as also to turn the greatest number of coefficients B^m_{ln} into constants—this path may be called direct canonization. If free parametric derivatives remain in B^m_{ln}, then the problem of constructing the group G_r is solved by prolonging the system (3*); in this case the space $T(y^\alpha)$ is extended to the space $V(\xi^\alpha)$.

As a concrete example, let us investigate by É. Cartan’ s method the equations describing one-dimensional gas flows with variable cross-sectional area of the flow tube ([8]):

$$\begin{gathered} u_t + uu_x - p_x/\rho = 0, \qquad p_t + up_x + Au_x + uAF_x = 0, \\ \rho_t + u\rho_x + \rho u_x + u\rho F_x = 0, \end{gathered} \tag{1}$$

where $A(p,\rho), F(x)$ are certain unspecified functions. We replace system (1) by a system of Pfaff forms, supplementing them with two further forms, linear in dx, dt, up to a complete basis $T(x,t,u,p,\rho)$:

$$\tilde\omega^6 = du - u_{xdx} + (uu_x + p_x/\rho)dt;$$

$$\tilde{\omega}^7 = dp - p_{xdx} + (up_x + Au_x + uAF_x)dt;$$
$$\tilde{\omega}^8 = d\rho - \rho_{xdx} + (u\rho_x + \rho u_x + u\rho F_x)dt;$$
$$\tilde{\omega}^1 = dx - (u-a)dt; \qquad \tilde{\omega}^2 = dx - (u+a)dt,$$

and, for convenience, pass to the forms

$$\omega^6 = a\rho\tilde{\omega}^6 - \tilde{\omega}^7; \quad \omega^7 = a\rho\tilde{\omega}^6 + \tilde{\omega}^7; \quad \omega^8 = \rho\tilde{\omega}^7 - A\tilde{\omega}^8; \quad \omega^1 = \tilde{\omega}^1; \quad \omega^2 = \tilde{\omega}^2.$$

Here $a = \sqrt{A/\rho}$. On solutions of system (1) the forms ω^6—ω^8, naturally, vanish. Differentiating the forms ω^i exteriorly, we write the system of exterior differential equations as follows:

$$\begin{aligned}
D\omega^1 &= \omega\wedge\omega^1 + b^1_{16}\omega^1\wedge\omega^6 + b^1_{26}\omega^2\wedge\omega^6 + b^1_{27}\omega^2\wedge\omega^7 + b^1_{28}\omega^2\wedge\omega^8;\\
D\omega^2 &= \omega\wedge\omega^2 - b^1_{27}\omega^1\wedge\omega^6 - b^1_{26}\omega^1\wedge\omega^7 - b^1_{16}\omega^2\wedge\omega^7 + b^1_{28}\omega^1\wedge\omega^8;\\
D\omega^6 &= \omega^3\wedge\omega^1 + \theta^6_6\wedge\omega^6 + b^6_{27}\omega^2\wedge\omega^7 + b^6_{28}\omega^2\wedge\omega^8 + b^6_{78}\omega^7\wedge\omega^8;\\
D\omega^7 &= \omega^4\wedge\omega^2 + \theta^7_7\wedge\omega^7 + b^7_{16}\omega^1\wedge\omega^6 + b^7_{18}\omega^1\wedge\omega^8 + b^7_{68}\omega^6\wedge\omega^8;\\
D\omega^8 &= \omega^5\wedge(\omega^1-\omega^2) + \theta^8_8\wedge\omega^8 + b^8_6(\omega^1-\omega^2)\wedge\omega^6 + b^8_7(\omega^1-\omega^2)\wedge\omega^7,
\end{aligned} \tag{2}$$

where $\omega, \theta^6_6, \theta^7_7, \theta^8_8$ are certain linear combinations of the forms ω^i; $\omega^3, \omega^4, \omega^5$ are linear combinations with respect to $du_x, dp_x, d\rho_x$ and the principal forms ω^i; the coefficients b^i_{jk} have the form:

$$\begin{aligned}
&b^1_{16} = -b^2_{27} = 1/4A; \qquad b^1_{26} = -b^2_{17} = -(A_p+\lambda+1)/8A;\\
&b^1_{27} = -b^2_{16} = (A_p+\lambda-3)/8A; \qquad b^1_{28} = b^2_{18} = (1-\lambda)/4\rho A;\\
&\lambda = \rho A_\rho/A; \qquad b^6_{78} = b^7_{68} = (1+\lambda)/\rho A;\\
&b^6_{27} = u_x(3-A_p-\lambda)/8a + p_x(A_p-4)/8A + \rho_x(\lambda+1)/8\rho +\\
&\quad + F_x(u-2a-uA_p-u\lambda)/8a;\\
&b^6_{28} = u_x(\lambda/2\rho a) + p_x(1/2\rho A) + F_x(u\lambda/2\rho a);\\
&b^7_{16} = u_x(A_p+\lambda-3)/8a + p_x(A_p-4)/8A + \rho_x(\lambda+1)/8\rho +\\
&\quad + F_x(uA_p+u\lambda-u-2a)/8a;\\
&b^7_{18} = u_x(\lambda/2\rho a) - p_x(1/2\rho A) + F_x(u\lambda/2\rho a);\\
&b^8_6 = b^8_7 = -p_x(1/4a^2) + \rho_x(1/4).
\end{aligned} \tag{3}$$

System (2) admits the transformation (we assume that $\lambda \neq 1$)

$$\overline{\omega}^6 = \omega^6/4A + \bar{l}\omega^1, \qquad \overline{\omega}^7 = \omega^7/4A + \overline{m}\omega^2,$$
$$\overline{\omega}^8 = (1-\lambda)\omega^8/4\rho A + \overline{n}(\omega^1+\omega^2), \tag{4}$$

where $\bar{l}, \overline{m}, \overline{n}$ are certain new variables which may be regarded as elements of the matrix A_l^m. The coefficients b_{jk}^i pass into $\bar{b}_{jk}^i$, where

$$\bar{b}_{16}^1 = 1; \qquad \bar{b}_{26}^1 = (A_p+\lambda+1)/2; \qquad \bar{b}_{27}^1 = (A_p+\lambda-3)/2;$$

$$\bar{b}_{27}^6 = b_{27}^6 + \bar{l}(A_p+\lambda+1)/2 + \overline{n}(1+\lambda)/(1-\lambda);$$

$$\bar{b}_{28}^6 = b_{28}^6\rho/(1-\lambda) + \bar{l} - \overline{n}(1+\lambda)/(1-\lambda);$$

$$\bar{b}_{78}^6 = \bar{b}_{68}^7 = (1+\lambda)/(1-\lambda);$$

$$\bar{b}_{18}^7 = b_{18}^7\rho/(1-\lambda) -$$

$$-\bar{l}(1+\lambda)/(1-\lambda) + \bar{m}; \qquad \bar{b}_{16}^7 = \bar{b}_{16}^7 - \bar{m}(A_p+\lambda-3)/2 + \bar{n}(1+$$
$$+\lambda)/(1-\lambda); \quad \bar{b}_6^8 = b_6^8(1-\lambda)/\rho + 2\bar{n}.$$

The forms $\omega, \theta_6^6, \theta_7^7, \theta_8^8, \omega^3, \omega^4, \omega^5$ will pass into some new forms $\bar{\omega}, \bar{\theta}_6^6, \dots, \bar{\omega}^5$. We now choose $\bar{l}, \bar{m}, \bar{n}$ so as to set equal to zero any 3 of the coefficients $\bar{b}_{jk}^i$, for example $\bar{b}_6^8, \bar{b}_{28}^7, \bar{b}_{18}^7$. For $\bar{l}, \bar{m}, \bar{n}$ one obtains a linear nonhomogeneous system W^1 with determinant $\Delta = 2[1-(1+\lambda)^2/(1-\lambda)^2]$. This determinant vanishes only for $\lambda = 0$, i.e., for $A_p = 0$. Assuming that $A_p \neq 0$, solving the system W^1 with respect to $\bar{l}, \bar{m}, \bar{n}$ and substituting in (4), we obtain

$$4\bar{\omega}^6 = du/a - dp/A + MF_x(dx - u\,dt);$$

$$4\bar{\omega}^7 = du/a + dp/A + MF_x(dx - u\,dt);$$
$$4\bar{\omega}^8 = (1-\lambda)(dp/A - d\rho/\rho); \qquad M = u/a.$$

The forms $\bar{\omega}^1 = \omega^1; \bar{\omega}^2 = \omega^2; \bar{\omega}^6, \dots, \bar{\omega}^8$ contain no parametric derivatives; therefore the system of exterior differential equations (for simplicity it is better to pass to the forms $\theta^1 = \omega^1+\omega^2; \theta^2 = \omega^1-\omega^2; \theta^6 = \bar{\omega}^7+\bar{\omega}^6; \theta^7 = \bar{\omega}^7-\bar{\omega}^6; \theta^8 = \bar{\omega}^8$) has the form

$$D\theta^i = C_{jk}^i\theta^j \wedge \theta^k,$$

where the coefficients C^i_{jk} are as follows:

$$C^1_{12} = MF_x/2; \qquad C^1_{26} = 2; \qquad C^2_{82} = 2; \qquad C^2_{72} = A_p + \lambda - 1;$$
$$C^6_{12} = M^2(F_x^2 - F_{xx})/8; \qquad C^6_{61} = F_x/2; \qquad C^6_{62} = -MF_x/2;$$
$$C^6_{76} = 1 - \lambda - A_p; \qquad C^6_{86} = -2; \qquad C^7_{87} = 4\lambda/(1-\lambda);$$
$$C^8_{78} = 2[(A\lambda_p + \rho\lambda_\rho)/(\lambda - 1) - \lambda]. \tag{5}$$

(Recall that here we assume $\lambda \neq 1$.) For $\lambda = 0$ the coefficients $\bar{b}^6_{27}, \bar{b}^6_{28}, \bar{b}^7_{16}$ can be canonized. The determinant Δ will then be equal to $(3 - A_p)$ and, consequently, will be nonzero if $A \neq 3p$, which we shall assume. The coefficients C^i_{jk} have the form (we again pass to the forms θ^i):

$$C^1_{12} = \alpha MF_x/2; \qquad C^1_{62} = -2; \qquad C^2_{12} = -F_x/2; \qquad C^2_{72} = A_p - 1;$$
$$C^2_{82} = 2; \qquad 2C^6_{12} = M^2[\alpha(\alpha F_x^2 - F_{xx})/4 - AA_{pp}F_x^2/(3 - A_p)^2];$$
$$C^6_{61} = -F_x/(3 - A_p); \qquad C^6_{71} = -MAA_{pp}F_x/(3 - A_p)^2;$$
$$C^6_{67} = A_p - 1; \qquad C^6_{68} = 2; \qquad C^7_{12} = -M(\alpha F_x^2 - F_{xx})/(3 - A_p); \tag{6}$$
$$C^7_{62} = F_x/(3 - A_p); \qquad C^7_{72} = MAA_{pp}F_x/(3 - A_p)^2;$$
$$C^8_{12} = (\alpha F_x^2 - F_{xx})M/8; \qquad C^8_{62} = -F_x/2; \qquad \alpha = (1 - A_p)/(3 - A_p).$$

The case $\lambda = 1$ also, generally speaking, is special, since $\Delta = \infty$. For $\lambda = 1$ it is simpler to use direct canonization, choosing the parametric derivatives u_x, p_x, ρ_x from the conditions that the coefficients $b^6_{28}, b^7_{18}, b^8_6$ vanish. This immediately gives $p_x = \rho_x = 0$; $u_x = -uF_x$. Substituting these values into the forms $\omega^6, \omega^7, \omega^8$ and, for convenience, passing to the forms $\theta^1 = \omega^1 + \omega^2; \theta^2 = \omega^1 - \omega^2; \theta^6 = (\omega^7 + \omega^6)/4A; \theta^7 = (\omega^7 - \omega^6)/4A; \theta^8 = \omega^8/2\rho a$, we obtain the structural coefficients:

$$C^1_{12} = MF_x/2; \qquad C^1_{62} = -1; \qquad C^2_{72} = A_p; \qquad C^6_{12} = M^2(F_x^2 - F_{xx})/8; \tag{7}$$
$$C^6_{61} = F_x/2; \qquad C^6_{62} = MF_x/2; \qquad C^6_{76} = -A_p; \qquad C^7_{87} = C^8_{87} = 2.$$

We see that in all cases the coefficients C^i_{jk} depend on the functions $A(p, \rho), F(x)$ and their derivatives. Special from the point of view of canonization is the value $A(p, \rho)$ satisfying the equation $A_p = 0$ ([8]).

The presence of a complete set of coefficients C^i_{jk} makes it possible to consider problems connected with the investigation of the group G_r admitted by the initial system of equations, in the present case by system (1). One of such interesting and important problems is the problem of classification with respect to the functions ([4]), connected with the construction of systems of intransitivity. The solution of this problem makes it possible to divide the functions F^σ into classes, for each of which its own invariant group is constructed.

It is known that a system of forms ω^i satisfying the equations

$$D\omega^i = C^i_{jk}(\xi)\omega^j \wedge \omega^k$$

defines a group if and only if the dif-

differentials dC^i_{jk} are expanded in invariant forms ω^i with coefficients depending only on $\overline{C}^i_{jk}$, where $\overline{C}^i_{jk}$ are functionally independent of C^i_{jk}, i.e., the condition $dC^i_{jk} = E^i_{jkl}(C^i_{jk})\omega^l$ must be satisfied.

We restrict ourselves to consideration of the case $F(x) = 0$, $A = A(p, \rho)$, which is sufficiently general; moreover, we shall assume that $\lambda \neq 1; 0$. Considering the coefficients C^i_{jk} from (5), we see that as $\overline{C}^i_{jk}$ one may take $\lambda = \rho A_\rho / A$, $\mu = A_p$. The differentials $d\lambda, d\mu$ in the expansion in the forms θ^i are

$$d\lambda = 2\theta^7(A\lambda_p + \rho\lambda_\rho) + 4\rho\lambda_\rho\theta^8/(\lambda - 1);$$

$$d\mu = 2\theta^7(AA_{\rho p} + \rho A_{pp}) + 4\rho A_{pp}\theta^8/(\lambda - 1). \tag{8}$$

If p, ρ are invariants, then $A(p, \rho)$ may be arbitrary, and in the representation space of the group with invariant forms ω^i three-dimensional systems of intransitivity will be distinguished.

However, $A(p, \rho)$ can be chosen so that λ, μ and the coefficients of the forms θ^7, θ^8 in system (8) depend only on one variable $\eta = \eta(p, \rho)$. In this case $\lambda, \mu, \rho\lambda_\rho, A\lambda_p, AA_{pp}$ will be functionally dependent. This leads to the following equations for $A(p, \rho)$:

$$A_{pp}(A_\rho + \rho A_{\rho\rho} - \lambda A_\rho) = \rho A_{p\rho}(A_{pp} - \lambda\mu/\rho); \tag{9a}$$

$$A_{pp}(2A_{\rho\rho} + \rho A_{\rho\rho\rho} - \lambda A_{\rho\rho}) = \rho A_{p\rho}(A_{p\rho} - \mu A_{\rho\rho}/A); \tag{9b}$$

$$A_{pp}(A_{pp} + \rho A_{pp\rho}) = \rho A_{p\rho}A_{ppp}; \tag{9c}$$

$$A_{pp}(A_\rho A_{pp} + AA_{pp\rho}) = A_{p\rho}(A_{pA_{pp}} + AA_{ppp}). \tag{9d}$$

Investigation of the compatibility of this system of equations gives those forms of the function $A(p, \rho)$ for which $\lambda, \mu, \rho\lambda_\rho, A\lambda_p, AA_{pp}$ will depend only on $\eta(p, \rho)$. For such $A(p, \rho)$ we obtain four-dimensional systems of intransitivity.

Let us consider several specific forms of the function $A(p, \rho)$; for definiteness we shall assume that $A_{pp} \neq 0$, $A_{pp} \neq 0$. In this case, from (9c) it immediately follows that $A_p = \mu = f(\rho e^{\int dp/k(p)})$, i.e. $\eta = \rho e^{\int dp/k(p)}$, where f, k are arbitrary functions of their arguments. The condition that $d\eta(p, \rho)$, in its expansion in the forms θ^7, θ^8, have coefficients depending only on η leads to $A/k(p) = \varphi(\eta)$; $\varphi(\eta)$ is an arbitrary function. Differentiating this relation with respect to p, we arrive at the condition $k' = \text{const}$, i.e. $k(p) = k_0 p + k_1$ $(k_0, k_1 = \text{const})$. For $k_0 \neq 0$ this gives $A(p, \rho) = (k_0 p + k_1)\varphi[\rho(k_0 p + k_1)^{1/k_0}]$; for $k_0 = 0$ we have $A(p, \rho) = \varphi(\rho e^{p/k_1})$ ([1]). The cases $A_{pp} = 0$; $A_{p\rho} \neq 0$, and others are considered in the same way; for them four-dimensional systems of intransitivity are obtained. For $\lambda = 0, 1$ the consideration is analogous.

Five-dimensional systems of intransitivity are obtained from the requirement that all coefficients C^i_{jk} from (5) be converted into constants. Requiring the conversion into a constant of C^7_{87}, we obtain $A = k(p)\rho^m$, (i.e. $\lambda = m$), where $k(p)$ is an arbitrary function of p. Now the condition of constancy of the coefficient C^2_{72} leads to the requirement $k(p) = \text{const}$. Consequently, five-dimensional systems of intransitivity are possible only for $A = k\rho^m$ $(m \neq 0, 1)$.

In conclusion I consider it my pleasant duty to express my deep gratitude to A. M. Vasil' ev for his attention to the present work.

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Note: Figure translations are in progress. See original paper for figures.