UNIFORM APPROXIMATIONS BY RATIONAL FUNCTIONS (ALGEBRAIC AND TRIGONOMETRIC) AND GLOBAL FUNCTIONAL PROPERTIES

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UNIFORM APPROXIMATIONS BY RATIONAL FUNCTIONS (ALGEBRAIC AND TRIGONOMETRIC) AND GLOBAL FUNCTIONAL PROPERTIES

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Without dwelling on the history of the question (it may be found in (5)), let us recall some definitions and notation.

Let $\Phi(u)$ be increasing, continuous, and convex downward on $[0, \infty)$, $\Phi(0) = 0$. We say that Φ has the Δ_2-property, $\Phi \in \Delta_2$, if for all $u \geq 0$ we have $\Phi(2u) \leq C\Phi(u)$, where $C = C(\Phi) = \text{const}$. The Φ-variation of a function $f(x)$ $(x \in [a,b], -\infty \leq a < b \leq +\infty)$ is defined as follows (see (1)):

$$V_\Phi(f) = V_\Phi(f, [a,b]) = \sup\left\{\sum_{k=1}^{n} \Phi\left(|f(x_k) - f(x_{k-1})|\right)\right\}, \tag{1}$$

where the least upper bound is taken over all $\{x_k\} : a = x_0 < x_1 < \cdots < x_n = b,\ n = 1, 2, \ldots$.

If $V_\Phi(f) < \infty$, then $f \in V_\Phi$. If $kf \in V_\Phi$ for some $k = \text{const} > 0$, then $f \in V^*_\Phi$. Obviously, $V_\Phi \subset V^*_\Phi$ and $V_\Phi = V^*_\Phi$ for $\Phi \in \Delta_2$. Note that any function f continuous on $[a,b]$ belongs to some V^*_Φ. If, following Musielak and Orlicz (1), one introduces the norm

$$\|f\|_\Phi = \|f\|_{\Phi,[a,b]} = \inf\{k : k > 0,\ V_\Phi(f/k) \leq 1\} \tag{2}$$

and regards $f_1 = f_2$ when $f_1(x) - f_2(x) \equiv \text{const}$, then V^*_Φ becomes a complete linear normed space (with the usual addition of functions and multiplication of a function by a number) ((1), Sec. 3.21), and, if $\|f - f_n\|_\Phi \to 0$, then $V_\Phi(k(f - f_n)) \to 0$ for some $k = \text{const} > 0$ ((1), Sec. 3.11).

A function $f(x)$ $(x \in [a,b])$ is called Φ-absolutely continuous, $f \in AC_\Phi$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum \Phi(|f(\beta_i) - f(\alpha_i)|) < \varepsilon$ whenever $\sum \Phi(\beta_i - \alpha_i) < \delta$, $a \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq b$. $f \in AC^*_\Phi$ if $kf \in AC_\Phi$

for some $k = \text{const} > 0$. Obviously, $AC_\Phi \subset V_\Phi$, $AC^*_\Phi \subset V^*_\Phi$, $AC_\Phi \subset AC^*_\Phi$, and $AC_\Phi = AC^*_\Phi$ for $\Phi \in \Delta_2$. AC^*_Φ is a closed subspace of the space V^*_Φ ((1), Sec. 3.21).

We say that $f \in L_\Phi[a,b] = L_\Phi$ if the function $\Phi(|f(x)|)$ is summable on $[a,b]$, and $f \in L^*_\Phi[a,b] = L^*_\Phi$ if $kf \in L_\Phi$ for some $k = \text{const} > 0$. The norm in L^*_Φ is defined as follows (see (3)):

$$\|f\|_{L_\Phi[a,b]} = \inf\left\{k : k > 0, \int \Phi(|f|/k)\,dx \le 1\right\}. \tag{3}$$

L^*_Φ (the Orlicz space) is a complete normed linear space. If $\Phi \in \Delta_2$, then $L_\Phi = L^*_\Phi$. If $\Phi(u) = u^p$, $p \ge 1$, then $L_\Phi = L^p$, and $\|f\|$ in the metric L^*_Φ coincides with $\|f\|$ in L^p.

* For example, with $\Phi(u) = \tilde\omega^{-1}(u/3)$, where $\tilde\omega$ is a regularization of the modulus of continuity of the function f; see (4), Sec. 3.3.

For $\delta > 0$ (and $f_h(x) = f(x+h)$) the quantity

$$\omega_\Phi(\delta, f) = \omega_\Phi(\delta, f[a,b]) = \sup_{|h|\le\delta} \|f - f_h\|_{L\Phi([a,b]\cap[a-h,b-h])} \tag{4}$$

will be called the Φ-modulus of continuity of the function f (on $[a,b]$). $R_n = R_n[f] = R_n[f,[a,b]]$ denotes the least deviation (in the uniform metric) of the function f from real rational functions (algebraic or trigonometric) of degree (order) not exceeding n* (see (5)); $r_n(x) = r_n(x,f)$ is a rational function of order $\le n$ for which $\max|f(x) - r_n(x)| = R_n$. In the case of rational trigonometric functions we always assume $0 \le a < b \le 2\pi$.

1°. **Rate of approximation and Φ-variation.**

Lemma 1. For every rational (algebraic or trigonometric) function $r(x)$ we have:

$$V_\Phi(r) \le 2n\Phi(2M) \tag{5a}$$

$$\|r\|_\Phi \le 4M/\Phi^{-1}(1/n), \tag{5b}$$

where n is the order of the function r, $M = \max\{|r(x)| : x \in [a,b]\}$.

Proof. From the increase of Φ it follows that the sum (1) will not decrease if to the partition points x_k one adds a point y, $x_{k-1} < y < x_k$, at which $r(y) \in [r(x_{k-1}), r(x_k)]$.

From the convexity downward of Φ it follows that this sum will not decrease if one removes a partition point x_k for which $r(x_k) \in [r(x_{k-1}), r(x_{k+1})]$.

Thus,

$$V_\Phi(r) = \sum \Phi(|r(y_i) - r(y_{i-1})|),$$

where the y_i are some of the points $y \in [a, b]$ at which r attains local extrema ($y_0 = a < y_1 < \cdots < y_k = b$, $k \le 2n$). Hence we obtain (5a). From this and (2) we have: $\|r\|_\Phi \le 2M/\Phi^{-1}(1/2n)$. Taking into account that Φ^{-1} is convex upward and increasing, we obtain (5b).

Remark. The preceding arguments show that if the graph of the function $y = f(x)$ can be divided into n intervals of monotonicity, then estimate (5) remains valid ($r = f$). If, in addition, f is continuous, then the right-hand sides in inequalities (5) should be halved.

Theorem 1. Always

$$\|f\|_\Phi \le 100 \sum_{n=0}^{\infty} \lambda_{nR} n, \quad \text{where} \quad \lambda_{n-1} = \left[n\Phi^{-1}\left(\frac{1}{n}\right)\right]^{-1} \le [\Phi^{-1}(1)]^{-1}. \qquad (6)$$

Proof. Obviously,

$$\|f\|_\Phi \le \|r_1 - r_0\|_\Phi + \sum_{k=1}^{\infty} \|r_{2^k} - r_{2^{k-1}}\|_\Phi.$$

By Lemma 1,

$$\|r_1 - r_0\|_\Phi \le 8R_0/\Phi^{-1}(1);$$

$$\|r_{2^k} - r_{2^{k-1}}\|_\Phi \le 4 \cdot 2R_{2^{k-1}}/\Phi^{-1}(1/3 \cdot 2^{k-1}) \le$$

$$\le 48R_{2^{k-1}}/\Phi^{-1}(2^{-(k-2)}) \le 96 \sum_{2^{k-2}+1}^{2^{k-1}} R_{n-1} \left[n\Phi^{-1}\left(\frac{1}{n}\right)\right]^{-1}. \qquad (7)$$

Theorem 2. If

$$\sum R_n[f]\,[n\Phi^{-1}(1/n)]^{-1} < \infty,$$

then $f \in AC^*_\Phi$. If, moreover, $\rho_n(x)$ are rational functions of order respectively not exceeding n, for which (with some $C = \text{const}$)

$$\max\{|f(x)-\rho_n(x)| : x\in[a,b]\}\le CR_n[f],$$

then $\|f-\rho_n\|_\Phi \to 0$ and $V_\Phi(k(f-\rho_n))\to 0$ for some $k=\text{const}>0$ $(n\to\infty)$.

$*$ $r(x)=P(x)/Q(x)$ is a rational algebraic (trigonometric) function of order $\le n$, if P and Q are algebraic (respectively trigonometric) polynomials of order $\le n$.

Proof. Choose natural p from the condition $2^{p-1}\le n<2^p$. From the inequality

$$\|f-\rho_n\|_\Phi \le \sum_{p+1}^{\infty}\|\rho_{2^q}-\rho_{2^{q-1}}\|_\Phi + \|\rho_{2^p}-\rho_n\|_\Phi$$

and estimates of type (7), we find that $\|f-\rho_n\|_\Phi\to 0$, whence $f\in AC_\Phi^*$ (obviously, $\rho_n\in AC_\Phi$).

The case $\Phi(r)=r$ was considered by the author in (5). In this case the condition of the theorem cannot be weakened. The constructions preceding Theorem 2 of paper (5) show that, in the general case, if $a_0\ge a_1\ge\dots>0$ and $\sum\Phi(a_n)=\infty$, there always exists a function f, not belonging to V_Φ^*, for which $R_n[f]\le a_n$.

Thus, if $R_n[f]\le Cn^{-1/p-\varepsilon}$ $(p\ge 1;\ C,\varepsilon>0)$, then $f\in AC_\Phi$ with $\Phi(u)=u^p$.* However, the condition $R_n[f]\le Cn^{-1/p}$ is insufficient even for f to belong to V_Φ^*.

2°. Rate of approximation and the integral modulus of continuity.

Lemma 2a.** For any function $f(x)$ $(-\infty\le a\le x\le b\le+\infty)$, for $0<\delta<b-a$ we have:**

$$1)\quad \int_a^{b-\delta}\Phi(|f(x+\delta)-f(x)|)\,dx\le 2V_\Phi(f)\delta; \tag{8}$$

$$2)\quad \omega_\Phi(\delta,f)\le \Omega/\Phi^{-1}((b-a)^{-1}), \tag{9}$$

where Ω is the total oscillation of f on $[a,b]$; $\Omega=\max f-\min f$.

Proof. If $a,b\ne\pm\infty$, put $x_k=a+k\delta$ for $k=0,1,\dots,n-1=[(b-a)/\delta]$ and $x_n=b-\delta$. Let $\sum'$ and $\sum''$ denote, respectively, summation over odd and over even k. Then

$$\int_a^{b-\delta}\Phi(|f(x+\delta)-f(x)|)\,dx=\sum\int_{x_{k-1}}^{x_k}\le$$

$$\leq \left(\sum{}' + \sum{}''\right) \sup_{x_{k-1} \leq x \leq x_k} \Phi(|f(x+\delta) - f(x)|)\delta \leq 2V_\Phi(f)\delta,$$

and (8) is proved. The cases $a = -\infty$, $b = \infty$ are obtained by passage to the limit. Since always the integral in (8) is $\leq \Phi(\Omega)(b-a)$, (9) follows from (4) and (3). As a consequence of (8) and (5a) we obtain

Lemma 2. Let $r(x)$ be a rational (algebraic or trigonometric) function of order $\leq n$, whose modulus on $[a, b]$ does not exceed M $(-\infty \leq a < b \leq +\infty)$. Then

$$\int_a^{b-\delta} \Phi(|r(x+\delta) - r(x)|)\, dx \leq 4n\Phi(2M)\delta, \tag{10}$$

$$\omega_\Phi(\delta, r) \leq 8M/\Phi^{-1}(1/n\delta). \tag{11}$$

Remark. These estimates are, obviously, also valid for any functions $r = f$ (not necessarily continuous) of the kind discussed in the remark to Lemma 1.

Theorem 3. For any function $f(x)$ $(-\infty < a \leq x \leq b < \infty)$, the inequality holds

$$\omega_\Phi(\delta, f) \leq C \sum_{0<n<\delta^{-1}} \lambda_n(\delta) R_n, \qquad \lambda_{n-1}(\delta) = [n\Phi^{-1}(1/n\delta)]^{-1}, \tag{12}$$

where $C = C(\Phi, b-a)$.

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* It is easy to see that in this case $\|f\|_\Phi = \sup\left(\sum |f(x_i) - f(x_{i-1})|^p\right)^{1/p}$.

** This lemma generalizes Lemma 2 of B. I. Golubov' s paper (?), in which $\Phi(u) = u^p$, $p \geq 1$. Still earlier the case $\Phi(u) = u$ was noted by P. L. Ul' yanov.

Proof. Let $2^{-k-1} < \delta \leq 2^{-k}$. Obviously,

$$\omega_\Phi(\delta, f) \leq \omega_\Phi(\delta, f - r_{2^k}) + \sum_{p=1}^{k} \omega_\Phi(\delta, r_{2^p} - r_{2^{p-1}}) + \omega_\Phi(\delta, r_1 - r_0).$$

From (9) and (11) we have

$$\omega_\Phi(\delta, f - r_{2^k}) \leq \frac{2R_{2^k}}{\Phi^{-1}((b-a)^{-1})} \leq \frac{4\Phi^{-1}(4)}{\Phi^{-1}((b-a)^{-1})} \cdot 2^{k-1} \frac{R_{2^k}}{2^k\Phi^{-1}((\delta 2^{k-1})^{-1})},$$

$$\omega_\Phi(\delta, r_{2^p} - r_{2^{p-1}}) \le \frac{16 R_{2^{p-1}}}{\Phi^{-1}((6 \cdot 2^{p-2}\delta)^{-1})} \le 192 \, \frac{2^{p-2} R_{2^{p-1}}}{2^{p-1}\Phi^{-1}((2^{p-2}\delta)^{-1})}.$$

Hence we obtain (12).

Corollary. Let $\omega^{(1)}(\delta, f)$ be the modulus of continuity of the function $f(x)$ ($x \in [a,b]$) in the metric L ($L = L_\Phi$ for $\Phi(r) = r$). Then

$$\omega^{(1)}\left(\frac{1}{n}, f\right) \le C \cdot \frac{1}{n}(R_0[f] + R_1[f] + \cdots + R_{n-1}[f]),$$

where $C = 200(1 + b - a)$. In particular, if $R_n[f] = O(n^{-\alpha})$ $(0 < \alpha < 1)$, then f satisfies on $[a,b]$ the Lipschitz-Hölder integral condition of order α: $f \in \mathrm{Lip}(\alpha, 1)$.

This theorem is close in form to the theorems of S. N. Bernstein, A. F. Timan, and S. B. Stechkin (see ([6]), p. 344) for approximations by trigonometric polynomials.

Remarks.

1. The results of this paper remain valid when the interval $[a,b]$ is replaced by an arbitrary measurable set $E \subset (-\infty, \infty)$ (cf. ([5])), (if instead of the boundedness of $[a,b]$ one requires the boundedness of E).
2. Theorems 1-3 remain valid if by r_n, ρ_n one understands functions whose graphs are polygonal lines with the number of links $\le n$, or if one understands polynomials in a Haar, Walsh, etc., system (see the remarks to Lemmas 1, 2), and by R_n the corresponding deviations from f.
3. Let $E_n[f]$ be the least deviation of f from polynomials (algebraic or trigonometric) of degree $\le n$, and let Φ be as at the beginning of this paper. Then for every Φ there exists $f(x)$ ($x \in [0,1]$) with the following properties:

a) $R_n[f] \le E_n[f] \le C R_n[f]$ ($C = \text{const}, \ n \ge 0$);

b) $f \in V_\Phi^*$, but $f \notin V_\Psi^*$, if $\Psi(u)$ grows near zero essentially more slowly than $\Phi(u)$ (i.e., $\Psi(u)/\Phi(ku) \to \infty$ for every fixed $k > 0$ as $u \to 0$; this guarantees the strict inclusion $V_\Psi^* \subset V_\Phi^*$).

Thus the classes V_Φ^* capture something characteristic precisely of rational (and not polynomial) approximations. By substituting $\omega(\delta) = \Phi^{-1}(\delta)$, this assertion follows from the following fundamental result:

Theorem 4. For every modulus of continuity $\omega(\delta)$* there exists a function $f(x)$ with the properties:
a) its ordinary modulus of continuity $\omega(\delta, f)$ has order $\omega(\delta)$:

$$C_1\omega(\delta) \le \omega(\delta, f) \le C_2\omega(\delta);$$

b)

$$R_n[f] \le E_n[f] \le CR_n[f], \qquad R_{2.9^k}[f] = E_{2.9^k}[f]$$

$$(C,\ C_1,\ C_2 \text{ are constants } > 0;\ n,\ k = 0, 1, ...).$$

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* $\omega(\delta)$ is a modulus of continuity if $0 = \omega(+0) \le \omega(\delta) \le \omega(\delta{+}\eta) \le \omega(\delta){+}\omega(\eta)$ for all $\delta, \eta \ge 0$.

Note: Figure translations are in progress. See original paper for figures.