THE FIRST BOUNDARY-VALUE PROBLEM

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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM

FOR A CLASS OF EQUATIONS AND SYSTEMS

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At the present time, with the help of methods of functional analysis, equations and systems naturally connected with S. L. Sobolev spaces with weight or without weight have been studied quite fully. These questions are treated, for example, in the works ([1−9]). In the works ([4,9]) spaces were considered whose metric does not contain all derivatives of the highest order. In the present note we consider systems of equations (in the particular case, simply equations) and the spaces connected with them, which in the general case do not reduce to S. L. Sobolev spaces with weight or without weight; moreover, in their metric, along with derivatives, there occur differential expressions of a certain order. We shall need the following definitions for what follows.

Definition 1. We shall say that the vector-function $u(x) = (u_1(x), \dots, u_M(x))$ has in Ω, in the generalized sense of S. L. Sobolev, a differential expression

$$\omega = \mathcal{L}u = \sum_{j=1}^{M} \mathcal{L}^{(j)},$$

where $\mathcal{L}^{(j)}$ is some differential operator with constant coefficients, if $\omega, u_1, \dots, u_M$ are locally summable and for any $\vec{v}(x) \in C_0^{(\infty)}(\Omega)$

$$\int_\Omega \left(\omega v - \sum_{j=1}^{M} u_j \mathcal{L}^{(j)*} v \right) d\Omega = 0. \tag{1}$$

In what follows, identically zero differential expressions are not excluded from consideration.

Definition 2. For two real functions $f(z)$ and $g(z)$, defined on a set Z, we shall write $f(z) \sim g(z)$ on $G \subset Z$ if there exist two finite positive constants $C_1(G)$ and $C_2(G)$ such that

$$C_1(G) f(z) \le g(z) \le C_2(G) f(z)$$

for all $z \in G$.

Below E_n denotes the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$; instead of "open set" we shall write o.m.

1. Let Ω be a domain in E_n, $\mathfrak{A}_\sigma = \{\Omega_\lambda^{(\sigma)}\}$ and $\widetilde{\mathfrak{A}}_\sigma = \{\widetilde{\Omega}_\lambda^{(\sigma)}\}$ be systems of o.m.

$$(\sigma = 1, 2, \dots, N; \quad \lambda = 1, 2, \dots), \qquad \Omega \supset \Omega_\lambda^{(\sigma)} \supset \widetilde{\Omega}_\lambda^{(\sigma)}, \quad \operatorname{mes}(\Omega_\lambda^{(\sigma)} \setminus \widetilde{\Omega}_\lambda^{(\sigma)}) = 0,$$

$$\widetilde{\Omega}_\lambda^{(\sigma)} \subset \widetilde{\Omega}_1^{(\sigma)}, \qquad \Phi_{\sigma,\lambda} = \bigcup_{\sigma' < \sigma} \Omega_\lambda^{(\sigma')}.$$

Suppose that, for every σ, the following conditions are satisfied:
a) $\widetilde{\Omega}_1^{(\sigma)}$ is mapped one-to-one onto some o.m. in the space of variables $y = (y_1, y_2, \dots, y_n)$, and this mapping is continuously differentiable in both directions. Put

$$I_\sigma = |D(x_1, \dots, x_n)/D(y_1, \dots, y_n)|;$$

b)

$$\sum_{i=1}^{n} |\partial x_i / \partial y_n| \le c(\lambda)$$

and

$$|D_x^\alpha y_n(x)| \le c(\lambda) |y_n|^{1-|\alpha|}$$

on $\widetilde{\Omega}_1^{(\sigma)} \setminus \Phi_{\sigma,\lambda}$ for any λ and $|\alpha| \le m_0$;

c) $y_n(x)$ can be extended to $\Omega_1^{(\sigma)}$ in such a way that there it will have all continuous derivatives up to order m_0 inclusive; d) under the mapping $\widetilde{\Omega}_\lambda^{(\sigma)}$ indicated above, it passes to a set of the form $\nabla_\sigma \times J_\lambda^{(\sigma)}$, where ∇_σ is an open set in the variables $y' = (y_1, \dots, y_{n-1})$, and $J_\lambda^{(\sigma)}$ is an interval on the y_n-axis; e) either $J_\lambda^{(\sigma)} = (0, \rho_\lambda^{(\sigma)})$, where $\rho_\lambda^{(\sigma)} > 0$ and $\lim_{\lambda\to\infty} \rho_\lambda^{(\sigma)} = 0$, or $J_\lambda^{(\sigma)} = (\rho_\lambda^{(\sigma)}, \infty)$, where $0 < \rho_\lambda^{(\sigma)} < \infty$ and $\lim_{\lambda\to\infty} \rho_\lambda^{(\sigma)} = \infty$; in the first case we shall write $\mathfrak{A}_\sigma \to 0$, in the second $\mathfrak{A}_\sigma \to \infty$; f) for every $y_0' \in \nabla_\sigma$ there exists $\tilde{\lambda}(y_0')$ such that $\{x : \ y' = y_0'\} \cap \Phi_{\sigma,\lambda}$ is empty for $\lambda > \tilde{\lambda}(y_0')$; g) either $\Omega_{\lambda_2}^{(\sigma)} \supset (\overline{\Omega}_{\lambda_1}^{(\sigma)} \cap \Omega)$ for $\lambda_2 < \lambda_1$, or $\rho(\Phi_{\sigma,\lambda+1}, \Omega \setminus \Phi_{\sigma,\lambda}) > 0$ and there exists $\lambda_1(\lambda)$ such that $\Omega_{\lambda_2}^{(\sigma)} \setminus \overline{\Phi}_{\sigma,\lambda+1} \supset (\overline{\Omega}_{\lambda_3}^{(\sigma)} \cap \Omega) \setminus \overline{\Phi}_{\sigma,\lambda+1}$ for $\lambda_3 > \lambda_2 > \lambda_1$; h) for every λ there is a measurable set $\nabla_\sigma' \subset \nabla_\sigma$ and a $\lambda_1(\lambda)$ such that $(\nabla_\sigma' \times J_{\lambda_1}^{(\sigma)}) \cap \Phi_{\sigma,\lambda+1}$ is empty and $(\Omega \setminus \Phi_{\sigma,\lambda}) \cap (\nabla_\sigma \times J_{\lambda_1}^{(\sigma)}) \subset \nabla_\sigma' \times J_{\lambda_1}^{(\sigma)}$; i) $\Omega \setminus \bigcup_\sigma \Omega_\lambda^{(\sigma)}$ is compact in Ω for every λ.

Example 1. $n \leqslant 3$, $\Omega = \{x : \ x_n > 0\}$, $\Omega_\lambda^{(1)} = \widetilde{\Omega}_\lambda^{(1)} = \{x : \ |x| > \lambda\}$, $\Omega_\lambda^{(2)} = \widetilde{\Omega}_\lambda^{(2)} = \{x : \ 0 < x_n < 1/\lambda\}$; here $\mathfrak{A}_1 \to \infty$, $\mathfrak{A}_2 \to 0$.

Example 2. $n \geqslant 2$,

$$\Omega = \left\{ x : \ \sum_{i=1}^{n-1} x_i^2 < f^2(x_n), \ 0 < x_n < \infty \right\}, \quad f(x_n) > 0$$

on $(0,\infty)$ and $\sup_{x_n}|f'(x_n)|<\infty$, $\Omega_\lambda^{(2)}=\widetilde{\Omega}_\lambda^{(2)}=\{x:\ x\in\Omega,\ 0<x_n<1/\lambda\}$,

$$\Omega_\lambda^{(1)}=\widetilde{\Omega}_\lambda^{(1)}=\{x:\ x\in\Omega,\ x_n>\lambda\},$$

$$\Omega_\lambda^{(3)}=\left\{x:\ x\in\Omega,\ \frac{\lambda}{\lambda+1}f^2(x_n)<\sum_{i=1}^{n-1}x_i^2<f^2(x_n)\right\},$$

$$\widetilde{\Omega}_\lambda^{(3)}=\Omega_\lambda^{(3)}\cap\{x:\ 0<\varphi_1<\pi,\dots,0<\varphi_{n-3}<\pi,\ 0<\varphi_{n-2}<2\pi\}$$

$(i=1,\dots,n{-}3)$, where $(\varphi_1,\varphi_2,\dots,\varphi_{n-2})$ is a set of angular spherical coordinates. Here $\mathfrak{A}_1\to\infty$, $\mathfrak{A}_2\to 0$, $\mathfrak{A}_3\to 0$.

Example 3. The conditions a)–i) are also satisfied by bounded or unbounded domains having a simple boundary in the sense of S. L. Sobolev.

Definition 3. We shall say that $u(x)=(u_1(x),\dots,u_M(x))\in L$ if

$$g_\lambda^p(u)=\sum_T\sum_{\nu=1}^{\chi_T}\left[\int_\Omega b^{(T,\nu)}(x)\,|\mathcal{L}^{(T,\nu)}u|^{q(T,\nu)}\,dx\right]^{p/q(T,\nu)}+$$

$$+\sum_{j=1}^{M}\left(\sum_{l=1}^{N(j)}\left[\int_\Omega b^{(l,j)}(x)\sum_{|\alpha|=m(l,j)}|D^\alpha u_j|^{p(l,j)}\right]^{p/p(l,j)}\right)<\infty, \qquad (2)$$

where T denotes a certain set of pairs (l,j); $\sum_T$ denotes summation over all such sets;

$$\mathcal{L}^{(T,\nu)}u=\sum_{(l,j)\in T}\ \sum_{|\alpha|=m(l,j)+1}a_\alpha^{(l,j,T,\nu)}D^\alpha u_j;$$

all derivatives and differential expressions in (2) exist in the sense of S. L. Sobolev; $1<\min_{(T,\nu)}q^{(T,\nu)}$ and $\max_\nu q^{(T,\nu)}\leqslant\min_{(l,j)\in T}p^{(l,j)}$; $m_0\geqslant m(l,j)\geqslant 0$ for all l and j; $b^{(l,j)}(x)$ and $b^{(T,\nu)}(x)$ are continuous and positive functions in Ω, and, for any λ and σ,

$$b^{(l,j)}(x)I_\sigma\sim y_n^{\gamma_n(l,j,\sigma)}\Lambda^{(l,j,\sigma)}(y')$$

on the set $\widetilde{\Omega}_{\lambda_1}\setminus\Phi_{\sigma,\lambda}$ for some $\lambda_1(\lambda)$;

$$b^{(T,\lambda)}(x)y_n^{-q(T,\nu)}(x)[b^{(l,j)}(x)]^{-q(T,\nu)/p(l,j)}\in L_{\frac{p(l,j)}{p(l,j)-q(T,\nu)}}(\widetilde{\Omega}_1^{(\sigma)})$$

for all T, $\nu=1,\dots,\chi_T$, $(l,j)\in T$.

Introduce the following notation. In the case when $\mathfrak{A}_\sigma\to 0$, set

$$\omega^{(l,j,\sigma)}=\min\left(m^{(l,j)}-1,\ m^{(l,j)}-1-\left[\frac{\gamma^{(l,j,\sigma)}+1}{p^{(l,j)}}\right]\right).$$

In the case when $\mathfrak{A}_\sigma\to\infty$, set

$$\omega^{(l,j,\sigma)} = m^{(l,j)} + 1 - \frac{\gamma^{(l,j,\sigma)} + 1}{p^{(l,j)}},$$

if $\frac{\gamma^{(l,j,\sigma)} + 1}{p^{(l,j)}}$ is an integer and

$$1 \le \frac{\gamma^{(l,j,\sigma)} + 1}{p^{(l,j)}} \le m^{(l,j)};$$

otherwise

$$\omega^{(l,j,\sigma)} = m^{(l,j)} - \left[\frac{\gamma^{(l,j,\sigma)} + 1}{p^{(l,j)}}\right].$$

Definition 4. We shall say that $u(x) \in L^0$ if $u(x) \in L$ and
1) $\lim\limits_{y_n \to 0} D^\alpha u_j(x(y_n, y')) = 0$ for $|\alpha| \le \omega^{(l,j,\sigma)}$ and for almost all $y' \in \nabla_\sigma$, if $\mathfrak{A}_\sigma \to 0$;
2) $\lim\limits_{y_n \to \infty} D^\alpha u_j(x(y_n, y')) = 0$ for $\omega^{(l,j,\sigma)} \le |\alpha| \le m^{(l,j)} - 1$ and for almost all $y' \in \nabla_\sigma$, if $\mathfrak{A}_\sigma \to \infty$ $(j = 1, \dots, M;\ l = 1, \dots, N^{(j)};\ \sigma = 1, \dots, N)$.

Definition 5. We shall say that

$$u(x) = (u_1(x), \dots, u_M(x)) \in \vec{C}_0^{(\infty)}(\Omega),$$

if $u_j(x) \in C_0^{(\infty)}(\Omega)$ for all $j = 1, 2, \dots, M$.

Theorem 1. *Let either $N = 1$, or $N > 1$ and $\mathfrak{A}_\sigma \to 0$ for all $\sigma > 1$. Then $\vec{C}_0^{(\infty)}(\Omega)$ is dense in L^0 in the polynorm $g(u)$.*

Introduce the notation: $T_1 = \{\sigma : \mathfrak{A}_\sigma \to 0\}$, $T_2 = \{\sigma : \mathfrak{A}_\sigma \to \infty\}$.

Theorem 2. *Suppose that for each j $(j = 1, \dots, M)$ at least one of the following conditions is satisfied:*
1) T_1 is empty;
2) T_1 is nonempty and $\max\limits_l \left(\max\limits_{\sigma \in T_1} \omega^{(l,j,\sigma)}\right) < 0$;
3) T_1 and T_2 are nonempty and $\max\limits_l \left(0, \max\limits_{\sigma \in T_1} (\omega^{(l,j,\sigma)})\right) < 0$;
4) T_1 is nonempty and $\max\limits_l \left(0, \max(\omega^{(l,j,\sigma)} + 1)\right) \ge \min\limits_l m^{(l,j)}$. Then L^0 is complete in the polynorm $g(u)$.

2. In this section we shall consider applications of Theorems 1 and 2 to the solution of systems of differential equations. Write the polynorm $g(u)$ in the form

$$g^p(u) = \sum_{t=1}^{R} \left(\int_\Omega b^{(t)}(x)\, |\mathcal{L}^{(t)} u|^{q(t)}\, dx \right)^{p/q(t)},$$

where

$$\mathcal{L}^{(t)} u = \sum_{j=1}^{N} \mathcal{L}^{(t,j)} u_j,$$

$$\mathcal{L}^{(t,j)} u_j = \sum_{|\alpha|=m(t,j)} a_\alpha^{(t,j)} D^\alpha u_j,$$

and $a_\alpha^{(t,j)}$ are constants. Suppose, furthermore, that it is known that for vector-functions from L there exist generalized, in the sense of S. L. Sobolev, differential expressions

$$\mathcal{L}^{(R+1)}, \dots, \mathcal{L}^{(T)},$$

and on $\vec{C}_0^{(\infty)}(\Omega)$ the inequality

$$\int_\Omega b^{(t)}(x)\, |\mathcal{L}^{(t)} u|^{q(t)}\, dx \le g^{q(t)}(u)$$

holds for all $t = R+1, \dots, T$, where $b^{(t)}(x)$ is a certain continuous function on Ω, positive at every point $x \in \Omega$.

Consider the system of equations

$$\sum_{t=1}^{T} \mathcal{L}^{(t,j)*} \varphi_t(x, \mathcal{L}^{(1)} u, \dots, \mathcal{L}^{(R)} u) = f_j(x) \qquad (j = 1, \dots, M).$$

Definition 6. We shall say that $u(x) \in \widetilde{L}$ if

$$u(x) \in L \quad \text{and} \quad \int_\Omega b^{(t)}(x)\, |\mathcal{L}^{(t)} u|^{q(t)}\, dx < \infty$$

for $t = 1, 2, \dots, T$.

Definition 7. We shall say that $u(x)$ from $\widetilde{L}$ is a generalized solution of system (4), if

$$\int_\Omega \left(\sum_{t=1}^{T} \varphi_t \mathcal{L}^{(t)} v - \sum_{j=1}^{M} f_{jv} j \right) = 0$$

for every $v = (v_1(x), \dots, v_M(x)) \in \vec{C}_0^{(\infty)}(\Omega)$.

Theorem 3. Suppose: 1) the conditions of Theorems 1 and 2 are satisfied;
2) $\varphi_t(x, y_1, \dots, y_T)$ is continuous in the aggregate of its arguments on $\Omega \times E_T$;
3)

$$|\varphi_t(x, y_1, \dots, y_T)| \leq \tilde{c}\, b^{(t)\,1/q^{(t)\prime}} \left(\sum_{\gamma=1}^{T} [[unclear: coefficient]]^{(\gamma)-1/q^{(t)\prime}} |y_\gamma|^{q^{(\gamma)}/q^{(t)\prime}} \right)^T;$$

4) there exists a matrix

$$\psi_x \equiv \|\psi_{\nu\mu}(x)\| \quad (\nu, \mu = 1, \dots, M),$$

such that: a) the functions $\psi_{\nu\mu}(x)$ are infinitely differentiable; b) the determinant

$$|\varphi(x)| \neq 0$$

for all $x \in \Omega$; c)

$$\mathcal{L}^{(t)}(\psi u) = \sum_{j=1}^{T} a_{tj}(x)\, \mathcal{L}^{(j)} u$$

for all $u \in \vec{C}_0^{(\infty)}(\Omega)$ and $t = 1, \dots, R$, where

$$a_{tj} \equiv 0 \quad \text{if } q^{(t)} > q^{(j)},$$

and

$$|a_{tj}(x)|^{q^{(t)}} b^{(t)}(x)\, [b^{(j)}]^{-q^{(t)}/q^{(j)}} \in L_{(q^{(j)}/q^{(t)})'}(\Omega), \quad \text{if } q^{(t)} \leq q^{(j)};$$

2)

$$\sum_{t=1}^{T} (\varphi_t(x, y_1+\eta_1, \dots, y_T+\eta_T) - \varphi_t(x, y_1, \dots, y_T)) \times \sum_{j=1}^{T} a_{tj}(x)\eta_j \geq \sum_{t=1}^{R} b^{(t)}(x)|\eta_t|^{q^{(t)}}.$$

Then for any vector-function $\hat{u}(x) \in \tilde{L}$ and for any vector-function

$$f(x) = (f_1(x), \dots, f_M(x))$$

from L^0, there exists a generalized solution $u(x)$ of system (4) such that

$$u(x) - \hat{u}(x) \in L^0.$$

This solution is unique up to a vector-function from the set

$$H = \{u : u \in L^0,\ g(u) = 0\}.$$

Theorem 3 is proved on the basis of the results of F. Browder' s work ([8]).

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Note: Figure translations are in progress. See original paper for figures.