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Abstract

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ON THE SPECTRUM OF THE SCHRÖDINGER OPERATOR USED IN THE OPTICAL MODEL OF THE NUCLEUS

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In the problem of scattering of nucleons by nuclei (^{1}), the Schrödinger equation is considered

$$\mathcal{L}u - k^2u = -\Delta u - [k^2 - V(x) - T(r)(\bar{l}, \bar{s})]u = 0, \qquad k^2 > 0, \tag{1}$$

$$x = (x_1, x_2, x_3).$$

In this equation the wave function u is a two-component spinor, i.e., a quantity transformed according to the representation $D_{1/2}$ of the rotation group of three-dimensional Euclidean space (^{2}). The potential $V(x)$ and the function $T(r)$ are real functions tending to zero as $r = |x| \to \infty$. As a rule, one restricts oneself to the case of a spherically symmetric potential, but we shall consider the general case*. The operator $\bar{l}$ is the operator of orbital angular momentum. Its components have the form

$$l_x = i(\sin\varphi\,\partial/\partial\theta + \operatorname{ctg}\theta\cos\varphi\,\partial/\partial\varphi);$$

$$l_y = -i(\cos\varphi\,\partial/\partial\theta - \operatorname{ctg}\theta\sin\varphi\,\partial/\partial\varphi); \qquad l_z = -i\,\partial/\partial\varphi, \tag{2}$$

where r, θ, φ are the spherical coordinates of the point x.

The spin operator $\bar{s} = \bar{\sigma}/2$ has the following projections on the coordinate axes:

$$s_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad s_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

the term $T(r)(\bar{l}, \bar{s})$ takes into account the spin-orbit interaction of the scattered particles.

Introduce on the set of spinors

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

a Hilbert space H with finite norm

$$(u, u) = \|u\|^2 = \int [\,|u_1|^2 + |u_2|^2\,]\,dx, \tag{4}$$

where an integral without indicated limits of integration is taken over the whole three-dimensional space. The aim of the paper is a complete description of the spectrum of equation (1), the construction of eigenfunctions of the continuous spectrum for this equation, the proof of an expansion theorem for an arbitrary function from H in eigenfunctions, and the existence of the S-matrix and wave operators for the scattering problem under consideration. An analogous analysis for the Schrödinger operator without taking spin-orbit interaction into account in some infinite domains was carried out in (^{3}).

§ 1. We shall call a spinor u finite if outside some domain the functions $u_1(x)$ and $u_2(x)$ are equal to zero. Denote by D^0 the set of finite twice differentiable spinors.

Lemma 1. *The Schrödinger operator $\mathcal{L}$, considered on the set D^0, is symmetric and semibounded. Its closure is a self-adjoint operator.*

* Everything said below remains valid if $V(x)$ is a Hermitian matrix; in this case

$$|V(x)| \equiv \sum_{i,j=1}^{2} |V_{ij}(x)|.$$

Theorem 1. If $V(x)$ and $T(r)$ are real-valued functions, continuous outside and square-integrable inside some domain, tending to zero as $|x| \to \infty$,

$$\lim_{|x|\to\infty} \big[|V(x)| + |T(r)|\big] = 0, \tag{5}$$

then the negative spectrum of the self-adjoint operator $\mathcal{L}$ is discrete, and the positive spectrum is continuous.

If, moreover, the condition

$$\lim_{r\to\infty} \sup_{\theta,\varphi} r \int_r^\infty [|V(t,\theta,\varphi)| + |T(t)|^2]\,dt = 0, \tag{6}$$

is satisfied, then the negative spectrum of the operator $\mathcal{L}$ consists of a finite number of eigenvalues.

Theorem 2. If

$$\lim_{r\to\infty}\left[\sup_{|x|=r} r\,|V(x)|\right] < k; \qquad T(r) = 0 \quad \text{for } r > R_0, \tag{7}$$

then the operator $\mathcal{L}$ has no positive eigenvalues. In other words, under conditions (7), every square-integrable solution of equation (1) is identically zero.

Let $G(x, y, k)$ be the Green matrix for equation (1). Since the operator $T(r)(\bar{l}\bar{s})$ is a matrix differential operator, the Green function of equation (1) is a matrix

$$G(x,y,k) = \begin{pmatrix} G_{11}(x,y,k) & G_{12}(x,y,k) \\ G_{21}(x,y,k) & G_{22}(x,y,k) \end{pmatrix}. \tag{8}$$

It satisfies the matrix equation:

$$\mathcal{L}G = \delta(x-y)I, \tag{9}$$

where $\delta(x-y)$ is the delta function, and I is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

The function $G(x, y, k)$ admits analytic continuation into the half-plane $\operatorname{Im} k > 0$, i.e., all the functions $G_{ij}(x, y, k)$, $i, j = 1, 2$, have this property.

In what follows it is assumed that conditions (7) are satisfied and

$$|V(x)| < \frac{c}{1+|x|^{3+a}}, \qquad a > 0. \tag{11}$$

Theorem 3. The function $G(x, y, k)$ is a continuous function of the spectral parameter k in the half-plane $\operatorname{Im} k \geq 0$*.

It follows from Theorem 3 that

Corollary 1. The positive spectrum of the operator $\mathcal{L}$ is absolutely continuous. The formula holds

$$\frac{1}{\pi}\operatorname{Im} G(x,y,\sqrt{\lambda}) = d\theta(x,y,\lambda)/d\lambda, \qquad \lambda \geq 0. \tag{12}$$

Theorem 4. Let $|y| \to \infty$, with the point y receding to infinity along a ray whose direction is characterized by the unit vector $\bar{\omega} = \arg y$. Then

$$G(x,y,k) \underset{|y|\to\infty,\arg y=\bar{\omega}}{=} \frac{e^{iky}}{4\pi|y|}\,\Phi(x,\bar{\omega},k)(1+o(1)). \tag{13}$$

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* If the operator $\mathcal{L}$ has no negative discrete spectrum. In the contrary case, there exists a finite number of isolated simple poles lying on the positive part of the imaginary axis.

The matrix $\Phi(x,\bar{\omega},k)$ in our problem is analogous to the plane wave $e^{-ik(\omega,x)}$ in the simplest scalar problem ($T=V=0$). Note that

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|y|}}{|y|}\,e^{-ik(\bar{\omega}x)}(1+o(1)).$$

$$|y|\to\infty,\ \arg y=\bar{\omega}$$

Theorem 5 (on expansion in eigenfunctions). Let $u \subset H$. The dual formulas hold

$$u(x) = \frac{1}{(2\pi)^{3/2}}\int \Phi(x,\bar{\omega}\bar{k})\,\hat{u}(\bar{k})\,d\bar{k} + \sum_{p=1}^{n} c_p\varphi_p(x),$$

$$\hat{u}(\bar{k}) = \frac{1}{(2\pi)^{3/2}}\int \Phi^*(x,\bar{\omega},k)u(x)\,dx, \qquad c_p = (u,\varphi_p), \qquad \bar{k} = k\bar{\omega}, \tag{14}$$

where $\varphi_p(x)$ are eigenfunctions of the negative spectrum of the operator $\mathcal{L}$.

The columns of the matrix $\Phi(x,\bar{\omega},k)$ are eigenfunctions of the continuous spectrum of the operator $\mathcal{L}$.

p. 2. For the proof of Theorems 2–5, some results are needed which, it seems to us, are of independent interest. Consider the operator equation in an abstract Hilbert space H:

$$d^2v/dr^2 - Av/r^2 + (k^2-Q)v = 0, \qquad k^2>0. \tag{15}$$

In this equation r is a parameter, $r \geqslant 0$, A is a self-adjoint, unbounded, positive operator,

$$(Av,v)>0 \quad \text{for } \|v\|>0. \tag{16}$$

The operator Q has the form

$$Q = V(r) + T(r)R, \tag{17}$$

where $V(r)$ and $T(r)$ are bounded, continuous operators, and

$$\limsup_{r\to\infty} r\,\|V(r)\| < 2k\mu, \qquad \mu < \tfrac{1}{2}; \qquad \|T(r)\| = 0 \quad \text{for } r > R_0 = \text{const}. \tag{18}$$

The operator R is self-adjoint, possibly unbounded, for $r \leqslant R_0$.

Theorem 6. Under the assumptions made, every solution of equation (15) that is not identically zero satisfies the relations

$$\lim_{R\to\infty} \int_R^{R+b} r^{2\mu+\varepsilon} \|v\|^2\, dr > \text{const} > 0, \tag{19}$$

$$\liminf_{r\to\infty} r^{2\mu+\varepsilon} \left[\|v'\|^2 + k^2\|v\|^2\right] > 0 \tag{20}$$

for any $\varepsilon > 0$, any fixed $b > 0$.

If it is assumed that

$$\int_{R_0}^{\infty} \|V(r)\|\, dr < \infty, \tag{21}$$

then in formulas (19), (20) one may put $\varepsilon = \mu = 0$.

Remark 1. Assertions (19)-(20) are uniqueness theorems, important for a rigorous mathematical analysis of the spectrum of the Schrödinger operator. With their help Theorem 2 is proved. The results of p. 2 are obtained on the basis of a generalization of the methods of papers (^{3}, ^{4}).

p. 3. Let the assumptions of p. 1 be fulfilled and, in addition, let the functions $V(x)$ and $T(r)$ be finite, i.e. vanish outside some region.

Theorem 7. Under the assumptions made, the Green matrix-function $G(x, y, k)$ admits an analytic continuation to the entire plane of the complex variable k. This continuation is analytic for $\operatorname{Im} k \geqslant 0$* and meromorphic for $\operatorname{Im} k < 0$.

§ 4. Let the assumptions of § 1 be satisfied.

Theorem 8. In the scattering problem for the potential $V(x) + T(r)(\bar{l}, \bar{s})$, containing spin-orbit interaction, there exist wave operators $W_{\pm}$ and a unitary scattering operator S.

For the scalar problem in certain domains with an infinite boundary, an analogous theorem was proved in (3 ,)**.

§ 5. If the operator $\mathcal{L}$ has no discrete negative spectrum, then the limiting-amplitude principle is valid for this operator.

Theorem 9. Under the conditions of Theorem 7, the solution of the nonstationary problem

$$u_{tt} + \mathcal{L}u = f(x)e^{i\omega t}, \qquad |f(x)| < \frac{c}{1+|x|^{3+a}}, \quad a > 0, \tag{22}$$

$$u\big|_{t=0} = u_t\big|_{t=0} = 0 \tag{23}$$

admits the estimate, as $t \to \infty$,

$$u(x,t) = e^{i\omega t}v(x,\omega) + o(1), \tag{24}$$

where

$$\mathcal{L}v - \omega^2 v = f(x). \tag{25}$$

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* See the footnote to Theorem 3.

** In paper (3) there are a number of removable errors. In particular, in formula (2′), Δ should stand instead of ∇. In paper (3), in formula (8), D_x, D_y is printed; it should read D_x^2, D_y^2.