ON CONJUGATE FUNCTIONS

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Abstract

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MATHEMATICS

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ON CONJUGATE FUNCTIONS

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1. Let the 2π-periodic function $f(x)$ be summable on $[-\pi, \pi]$. Denote by $\sigma[f]$ the Fourier series of the function $f(x)$, and by $s_n(x, f)$ the partial sums of the series $\sigma[f]$. Further, as usual, by the symbol $\bar{f}(x)$ we shall denote the function conjugate to $f(x)$, i.e.

$$\bar{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\operatorname{ctg}\frac{t}{2}\,dt,$$

which exists almost everywhere for every summable (see (1), p. 528) function $f(x)$. It is known (see (1), p. 557) that if $f(x)$ is a bounded 2π-periodic function, then $\bar{f}(x)$ need not be bounded. However, as was shown by P. Turán (2) and M. Kinukawa (3), if $f(x)$ is an even 2π-periodic function bounded on $(-\infty, +\infty)$, then the functions

$$\varphi(x) = \frac{1}{\operatorname{tg} x/2}\int_0^x \bar{f}(t)\,dt, \qquad \psi(x) = \int_x^{\pi} \frac{\bar{f}(t)}{2\operatorname{tg} t/2}\,dt \quad (|x| \leqslant \pi)$$

are bounded on $(-\infty, +\infty)$*. The behavior of the partial sums of the series $\sigma[\varphi]$ and $\sigma[\psi]$ was studied by M. and S. Izumi (4). In particular, they showed that if $f(x)$ is an even bounded function and $\|s_n(x, f)\|_C = O(1)$, then $\|s_n(x,\varphi)\|_C = O(1)$, $\|s_n(x,\psi)\|_C = O(1)$. The question arises: what can be said about the continuity of the functions $\varphi(x)$ and $\psi(x)$, or about the uniform convergence of the series $\sigma[\varphi]$ and $\sigma[\psi]$?

Theorem 1. If $f(x)$ is an even 2π-periodic continuous function, then the function $\varphi(x)$ is also continuous. Moreover, if

$$f(0) = \int_0^{\pi} f(t)\,dt = 0,$$

then the function $\psi(x)$ is also continuous, and its modulus of continuity is

$$\omega(\delta, \psi) = O\left\{\omega(\delta, f) + \delta \int_\delta^1 \frac{\omega(t, f)}{t^2}\, dt\right\},$$

where $\omega(\delta, u)$ is the modulus of continuity of the function $u(x) \in C(-\pi, \pi)$.

Theorem 2. Let $f(x)$ be an even 2π-periodic continuous function and $|s_n(0, f)| \leqslant M$ $(n = 1, 2, \ldots)$. Then the series $\sigma[f]$ converges uniformly. Moreover, if

$$f(0) = \int_0^\pi f(t)\, dt = 0,$$

then the series $\sigma[\psi]$ also converges uniformly.

Analogous assertions for the functions $\varphi(x)$ and $\psi(x)$ in the case when $f(x)$ is odd are, generally speaking, false.

2. Let us now consider a function of two variables $f(x, y)$. Suppose that it is periodic with respect to each of the variables and $f(x, y) \in$

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* Here it is assumed that the functions $\varphi(x)$ and $\psi(x)$ are extended periodically with period 2π from the interval $[-\pi, \pi]$ to the entire line.

$\in L(R)$, where $\bar{R} = [-\pi, \pi, -\pi, \pi]$. Consider the conjugate functions of two variables

$$\bar{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + s, y) \operatorname{ctg} \frac{s}{2}\, ds,$$

$$\bar{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y + t) \operatorname{ctg} \frac{t}{2}\, dt,$$

$$\bar{f}_3(x, y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) \operatorname{ctg} \frac{s}{2} \operatorname{ctg} \frac{t}{2}\, ds\, dt.$$

If $f(x, y)$ is an even function with respect to the aggregate of the two variables, i.e. $f(-x, -y) = f(x, y)$, then, generally speaking, the assertions of P. Turán $(^2)$ and M. Kinukawa $(^3)$ are false $(^5)$ for the functions $\bar{f}_i(x, y)$ $(i = 1, 2, 3)$.

Let us now suppose that $f(x, y)$ is an even function with respect to each variable, i.e. $f(-x, y) = f(x, -y) = f(x, y)$. Further, let

$$\varphi(x, y) = \frac{1}{\operatorname{tg} x/2 \operatorname{tg} y/2} \int_0^x \int_0^y \bar{f}_3(s, t)\, ds\, dt, \qquad \psi(x, y) = \int_x^\pi \int_y^\pi \frac{\bar{f}_3(s, t)}{4 \operatorname{tg} s/2 \operatorname{tg} t/2}\, ds\, dt,$$

$(x, y) \in R$, and

$$\varphi(x+2\pi, y) = \varphi(x, y+2\pi) = \varphi(x, y), \qquad \psi(x+2\pi, y) = \psi(x, y+2\pi) = \psi(x, y)$$

for all x, y.

Lemma 1. If $f(x, y)$ is a bounded function, even with respect to each variable, then the relations

$$\varphi(x, y) = \frac{1}{\pi^2 \operatorname{tg} x/2 \operatorname{tg} y/2} \int_0^\pi \int_0^\pi f(s, t) \log \left| \frac{\sin(x+s)/2 \sin(x-s)/2}{\sin^2 s/2} \right| \times \log \left| \frac{\sin(y+t)/2 \sin(y-t)/2}{\sin^2 t/2} \right| ds\, dt,$$

$$\begin{aligned} \psi(x, y) = \frac{1}{4\pi^2} \int_0^\pi \int_0^\pi f(s, t) \operatorname{ctg} \frac{s}{2} \operatorname{ctg} \frac{t}{2} \log \left| \frac{\sin(x+s)/2}{\sin(x-s)/2} \right| \log \left| \frac{\sin(y+t)/2}{\sin(y-t)/2} \right| ds\, dt \\ - \frac{\pi - x}{4\pi^2} \int_0^\pi \int_0^\pi f(s, t) \operatorname{ctg} \frac{t}{2} \log \left| \frac{\sin(y+t)/2}{\sin(y-t)/2} \right| ds\, dt \\ - \frac{\pi - y}{4\pi^2} \int_0^\pi \int_0^\pi f(s, t) \operatorname{ctg} \frac{s}{2} \log \left| \frac{\sin(x+s)/2}{\sin(x-s)/2} \right| ds\, dt \\ + \frac{(\pi - x)(\pi - y)}{16\pi^2} \int_0^\pi \int_0^\pi f(s, t)\, ds\, dt. \end{aligned}$$

hold.

On the basis of this lemma Theorems 3 and 4 are proved.

Theorem 3. If $f(x, y)$ is a bounded even function (with respect to each variable), then the functions $\varphi(x, y)$ and $\psi(x, y)$ are also bounded.

Theorem 4. If $f(x, y)$ is a continuous even function (with respect to each variable), then the function $\varphi(x, y)$ is continuous. Moreover, if

$$f(0, 0) = \int_0^\pi \int_0^\pi f(s, t)\, ds\, dt = 0,$$

then the function $\psi(x, y)$ is continuous and its modulus of continuity

$$\omega(\delta, \delta'; u) = O \left\{ \omega_1(\delta, f) + \omega_2(\delta', f) + \delta \int_\delta^1 \frac{\omega_1(s, f)}{s^2} ds + \delta' \int_{\delta'}^1 \frac{\omega_2(t, f)}{t^2} dt \right\},$$

where $\omega(\delta, \delta'; u)$, $\omega_1(\delta, u)$ and $\omega_2(\delta', u)$ are, respectively, the full and partial moduli of continuity of the function $u(x, y) \in C(R)$.

Analogous assertions are also valid for the functions

$$\bar{f}_i(x, y) \quad (i = 1, 2).$$

3. Let now $f(x, y) \in L(R)$ and

$$F(x, y) = \int_0^x \int_0^y f(s, t)\, ds\, dt.$$

Put

$$\Delta_x(F; x, y, s) = F(x + s, y) + F(x - s, y) - 2F(x, y),$$

$$\Delta_y(F; x, y, t) = F(x, y + t) + F(x, y - t) - 2F(x, y),$$

$$\Delta^2(F; x, y, s, t) = \Delta_x(\Delta_y(F; x, y, t)) = \Delta_y(\Delta_x(F; x, y, s));$$

$$\widetilde{F}_1(x, y) = \lim_{\varepsilon \to 0+} \int_\varepsilon^\pi \frac{\Delta_x(F; x, y, s)}{2 \sin^2 s/2}\, ds, \qquad \widetilde{F}_2(x, y) = \lim_{\eta \to 0+} \int_\eta^\pi \frac{\Delta_y(F; x, y, t)}{2 \sin^2 t/2}\, dt,$$

$$\widetilde{F}_3(x, y) = \lim_{\varepsilon, \eta \to 0+} \int_\varepsilon^\pi \int_\eta^\pi \frac{\Delta^2(F; x, y, s, t)}{4 \sin^2 s/2\, \sin^2 t/2}\, ds\, dt.$$

The question is: under what conditions do the functions $F_i(x, y)$ $(i = 1, 2, 3)$ exist, and what is the relation between them?

The functions $\widetilde{F}_i(x, y)$ $(i = 1, 2)$ exist almost everywhere for every summable 2π-periodic function $f(x, y)$, which may be obtained from the corresponding result of A. Plessner (^{6}). For the function $\widetilde{F}_3(x, y)$ the analogous assertion is, generally speaking, false, since the following holds.

Theorem 5. There exists a function $f(x, y) \in L(\log^+ L)^\alpha$ for all $\alpha \in [0, 1)$, for which almost everywhere on R

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^\pi \int_\varepsilon^\pi \frac{\Delta^2(F; x, y, s, t)}{4 \sin^2 s/2\, \sin t/2}\, ds\, dt$$

does not exist.

If, however, $f(x,y) \in L(\log^+ L)^\alpha$, $\alpha \geq 1$, then such an assertion is no longer valid, for the following is true.

Theorem 6. If $f(x,y) \in L \log^+ L$, then $\widetilde{F}_3(x,y)$ exists almost everywhere and, for almost all x and y,

$$\widetilde{F}_3(x,y) = \widetilde{F}_{i,j}(x,y) = \bar{f}_{i,j}(x,y) \qquad (i,j = 1,2;\ i \neq j)^*,$$

where

$$\widetilde{F}_{i,j}(x,y) = \lim_{\varepsilon \to 0+} \int_\varepsilon^\pi \frac{\Delta_x(\widetilde{F}_i; x, y, s)}{2\sin^2 s/2}\, ds \quad \text{for } i = 2,\ j = 1.$$

The results given here can be generalized also to the case of functions of n variables. We shall confine ourselves to considering the analogue of Theorem 6.

Let $f(x_1, \dots, x_n) \in L(R')$, where $R' = [-\pi, \pi; \dots; -\pi, \pi]$. Put

$$F(x_1, \dots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} f(s_1, \dots, s_n)\, ds_1 \cdots ds_n,$$

$$\Delta_k F \equiv \Delta_k(F; x_1, \dots, x_n, s_k) = F(x_1, \dots, x_k + s_k, \dots, x_n) +$$

$$+ F(x_1, \dots, x_k - s_k, \dots, x_n) - 2F(x_1, \dots, x_n),$$

$$\Delta^n(F; x_1, \dots, x_n; s_1, \dots, s_n) = \Delta_n(\Delta_{n-1}(\cdots(\Delta_1 F)) = \cdots = \Delta_1(\Delta_2(\cdots(\Delta_n F)\cdots)).$$

Theorem 7. Let $f(x_1, \dots, x_n) \in L(\log^+ L)^{n-1}$. Then almost everywhere on R' there exists

$$\lim_{(\varepsilon_1, \dots, \varepsilon_n) \to 0+} \int_{\varepsilon_1}^\pi \cdots \int_{\varepsilon_n}^\pi \frac{\Delta^n(F; x_1, \dots, x_n; s_1, \dots, s_n)}{2^n \sin^2 s_1/2 \dots \sin^2 s_n/2}\, ds_1 \dots ds_n,$$

and almost everywhere

$$\widetilde{F}_{2^n - 1}(x_1, \dots, x_n) = \widetilde{F}_{i_1, \dots, i_n}(x_1, \dots, x_n)$$

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* The symbol $\bar{f}_{i,j}(x,y)$ means that the conjugation operation is applied to the function $f(x,y)$ first in the i-th

$$(i_1, \dots, i_n = 1, 2, \dots, n, \text{ in the order } i_j \neq i_k),$$

where

$$\widetilde{F}_{i_k}(x_1, \dots, x_n) = \lim_{\varepsilon_{i_k} \to 0+} \int_{\varepsilon_{i_k}}^{\pi} \frac{\Delta_{i_k}(F; x_1, \dots, x_n, s_{i_k})}{2 \sin^2 s_{i_k}/2} \, ds_{i_k},$$

$$\widetilde{F}_{i_k, i_j}(x_1, \dots, x_n) = \lim_{\varepsilon_{i_j} \to 0+} \int_{\varepsilon_{i_j}}^{\pi} \frac{\Delta_{i_j}(\widetilde{F}_{i_k}; x_1, \dots, x_n, s_{i_j})}{2 \sin^2 s_{i_j}/2} \, ds_{i_j},$$

and the assertion loses its force if, in the hypothesis of the theorem, the exponent $n-1$ is replaced by a smaller one.

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Note: Figure translations are in progress. See original paper for figures.