POSITIVE MIXED PROBLEMS

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Abstract

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POSITIVE MIXED PROBLEMS

FOR CERTAIN HYPERBOLIC SYSTEMS

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In this note a mixed problem is considered in a cylindrical domain for a general first-order hyperbolic system, assumed to be symmetric (more precisely, Hermitian) only at the points of the lateral surface of the cylinder. An energy inequality and the existence of a strong solution are established under positivity assumptions on the boundary condition of the same kind as in $^{(1-3)}$.

Let G be a bounded domain in the real n-dimensional space R^n_x of points $x = (x^1, \dots, x^n)$, having an $(n-1)$-dimensional boundary Γ of smoothness class C^2. In the space $R^{n+1}_{t,x}$ of points (t, x) consider the cylindrical domain $\Omega = I \times G$, where I is the segment $\{t : 0 \le t \le T\}$. Denote by Ω' the lateral part $I \times \Gamma$ of the boundary of the domain Ω, by $G(\tau)$ the intersection of the domain Ω with the plane $t = \tau$, and by $\Gamma(\tau)$ the boundary of this intersection. By S denote the unit sphere $|\xi| = 1$ in the space R^n_ξ of points $\xi = (\xi_1, \dots, \xi_n)$.

Suppose that in the closure $\overline{\Omega}$ of the domain Ω there is given a system of first-order partial differential equations

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{\nu=1}^{n} A_\nu(t,x) \frac{\partial u}{\partial x^\nu} + C(t,x)u = f. \tag{1}$$

Here $u = u(t, x)$ and $f = f(t, x)$ are column vectors of height N with complex components; A_ν and C are square complex matrices of order N. We shall assume that $A_\nu \in C^2(\overline{\Omega})$.* It is sufficient to assume that the elements of the matrix C are bounded measurable functions.

We subject the solution $u(t, x)$ to the boundary condition

$$B(t,x)u(t,x) = g(t,x) \quad \text{on } \Omega', \tag{2}$$

where B is a smooth complex matrix on Ω', having N columns and a fixed number q of linearly independent rows, and g is a complex column vector of height q, given on Ω', and also to the initial condition

$$u(0, x) = h(x). \tag{3}$$

We assume system (1) to be hyperbolic in the generalized sense of I. G. Petrovskii $^{(5)}$ in Ω and Hermitian on Ω'. More precisely, let

$$a(t, x, \xi) = \sum_{\nu=1}^{n} \xi_\nu A_\nu(t, x). \tag{4}$$

We shall assume the following to hold.

Condition I. For each point $(t_0, x_0) \in \overline{\Omega}$ there exists a neighborhood U_0 in $R^{n+1}_{t,x}$, and for each point $\xi_0 \in S$ there exists a neighborhood S_0 on S, such that in

$$V_0 = (U_0 \cap \overline{\Omega}) \times S_0$$

there is defined a nonsingular matrix

$r(t, x, \xi)$ of order N and smoothness class $C^{2;\infty}_{t,x;\xi}(V_0)$*, possessing the following two properties: 1) the matrix rar^{-1} is diagonal and real in V_0 (it suffices to assume it Hermitian); 2) the matrix r is unitary for $(t, x) \in \Omega'$ (so that the A_ν are Hermitian on Ω').

Let us note two important cases in which Condition I is satisfied.

1°. The matrices A_ν are Hermitian everywhere in $\overline{\Omega}$, so that for r one may take the identity matrix. In this case the theorems formulated below are well known ($^{1-3,6,7}$).

2°. The system (1) is strictly hyperbolic in the sense of I. G. Petrovskii (^{5}) (i.e., the eigenvalues of the matrix (4) are real and pairwise distinct for $(t, x) \in \overline{\Omega}$, $\xi \neq 0$), and, moreover, the matrices A_ν are Hermitian on Ω'. In this case, for r^{-1} one must take the matrix whose columns are the normalized linearly independent eigenvectors of the matrix a. If, in addition, $n > 2$ and the domain G is simply connected, then r is immediately constructed globally (^{5}).

Put $a_l(t, x) = a(t, x, l)$, where $l = (l_1, \dots, l_n)$ is the unit vector of the inner normal to $\Gamma(t)$ at the point $(t, x) \in \Omega'$.

Condition II. $\det a_l(t, x) \neq 0$ on Ω'.

Examples show that there exist strictly hyperbolic systems with infinitely smooth $A_\nu(t, x)$, satisfying Conditions I and II and such that, for no nonsingular matrix $T(t, x)$, will the matrices $TA_\nu T^{-1}$ ($\nu = 1, \dots, n$) be simultaneously Hermitian in Ω.

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* Our smoothness assumptions can be weakened.

For two vectors $u = \{u_j\}$, $v = \{v_j\}$ from C^N put $u \cdot v = \sum u_j \bar{v}_j$. Denote by $\mathfrak{B}(t, x)$ the kernel of the matrix $B(t, x)$ in C^N. Following ($^{1-3}$), we formulate two more conditions.

Condition III. For every point $(t, x) \in \Omega'$ the quadratic form $u \cdot a_l(t, x)u$ is positive on $\mathfrak{B}(t, x)$:

$$u \cdot a_l(t, x)u \geq p_0 u \cdot u \quad \text{for} \quad 0 \neq u \in \mathfrak{B}(t, x), \tag{5}$$

where $p_0 = \text{const} > 0$.

Condition IV. Condition III is fulfilled, and, moreover, in any $(N - q + 1)$-dimensional subspace of C^N containing $\mathfrak{B}(t, x)$, there exists a vector v for which $v \cdot a_l(t, x)v < 0$.

Put

$$\|u\|^2_{G(t)} = \int_{G(t)} |u(t, x)|^2 \, dx, \qquad \|u\|^2_{\Gamma(t)} = \int_{\Gamma(t)} |u(t, x)|^2 \, dx',$$

where dx' is the surface element on $\Gamma(t)$. We define $\|g\|_{\Gamma(t)}$ and $\|h\|_G$ analogously.

Theorem 1. *Let Conditions I, II, and III be fulfilled. Then, for solutions* $u(t, x) \in C^1(\overline{\Omega})$ *of problem (1)–(3), the energy inequality*

$$\|u\|^2_{G(t)} + \int_0^t \|u\|^2_{\Gamma(\tau)} \, d\tau \leq \text{const} \left\{ \int_0^t \left[\|f\|^2_{G(\tau)} + \|g\|^2_{\Gamma(\tau)} \right] d\tau + \|h\|^2_G \right\}, \qquad 0 \leq t \leq T, \tag{6}$$

holds, where the constant does not depend on t *or on* $u(t, x)$.

In the proof, approaches proposed for other purposes in (^{4,8,9}) are used. After localization one computes the derivative

$$\frac{d}{dt} \int |R(t) u_\varepsilon(t, x)|^2 \, dx,$$

where $R(t)$ is a singular integral operator in R^n_x, depending on the parameter t, with symbol, roughly speaking, equal to r, and where $u_\varepsilon(t, x)$ is a smooth continuation of the function $u(t, x)$ to R^n_x, equal to 0 outside the ε-neighborhood of the domain $G(t)$; then one passes to the limit as $\varepsilon \to 0$.

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* The derivatives with respect to t, x of order ≤ 2 are continuous in (t, x, ξ) together with their derivatives of arbitrary order with respect to ξ.

Denote by $H(\Omega)$, $H^{(q)}(\Omega')$, and $H(G)$ the Hilbert spaces of (vector-) functions in Ω, Ω', and G, respectively, with norms defined by the equalities

$$\|u\|_\Omega^2 = \int_0^T \|u\|_{G(t)}^2 \, dt, \qquad \|g\|_{\Omega'}^2 = \int_0^T \|g\|_{\Gamma(t)}^2 \, dt,$$

and $\|w\|_G$; in the first and third cases the column vectors have height N, and in the second, height q. Let $f, u \in H(\Omega)$; $g \in H^{(q)}(\Omega')$; $v \in H^{(N)}(\Omega')$; $h, w \in H(G)$. We shall call a triple (u, v, w) a strong solution of problem (1)–(3) if there exists a sequence of functions $u_\nu(t,x) \in C^1(\bar{\Omega})$ such that, as $\nu \to \infty$,

$$\|u_\nu - u\|_\Omega + \|u_\nu - v\|_{\Omega'} + \|u_\nu(T,x) - w\|_G +$$

$$+ \|Lu_\nu - f\|_\Omega + \|Bu_\nu - g\|_{\Omega'} + \|u_\nu(0,x) - h\|_G \to 0. \tag{7}$$

Thus, for a strong solution, the values on Ω', $G(0)$, and $G(T)$ are defined.

Theorem 2. If conditions I, II, and IV are satisfied, then for any $f \in H(\Omega)$, $g \in H^{(q)}(\Omega')$, $h \in H(G)$ problem (1)–(3) has a strong solution (u, v, w). For strong solutions

$$\|u\|_\Omega + \|v\|_{\Omega'} + \|w\|_G \le \text{const}\{\|f\|_\Omega + \|g\|_{\Omega'} + \|h\|_G\}, \tag{8}$$

where the constant does not depend on f, g, or h.

The proof of existence is the same as in ($^{1-3}$): the adjoint problem is used and the theorem on coincidence of strong and weak solutions (^{2}) is applied.

Inequality (8) shows that the strong solution is unique and depends continuously on the right-hand sides.

Remarks. 1°. Theorems 1 and 2 remain valid if, with respect to the boundary condition on Ω', one assumes, as in ($^{1-3}$), that it can only be written locally in the form (2). In this case it is convenient to regard $g(t,x)$ as a section of some vector bundle over Ω' with a q-dimensional fiber. The norm $\|g\|_{\Omega'}$ is then defined by means of a partition of unity on Ω' (see, for example, (^{10}), Sec. 6). The corresponding definition of a strong solution is equivalent to that given in (3).

2°. Suppose that the matrices $A_\nu(t,x)$ are Hermitian not only on Ω', but also near Ω'. Then condition II may be replaced by the condition of constancy of the rank of the matrix $a_\nu(t,x)$ on Ω', and condition III by the condition of nonnegativity of the form $u \cdot a_\nu u$ on $\mathfrak{B}(t,x)$. Theorems 1 and 2 remain valid after, for example, the following modifications: $g = 0$; $\|u\|_{\Gamma(\tau)}^2$ on the left in (6) is replaced by $\int_{\Gamma(\tau)} u \cdot a_\nu u \, dx'$; in the definition of a strong solution

$Bu_\nu = 0$. Such theorems are obtained almost directly from the results of $({}^{1-3,8,9})$ by localization.

3°. Put, for an integer $l \geq 0$,

$$|||u|||^2_{G(t),\,l} = \sum_{\alpha+|\beta|\leq l} \int_{G(t)} \left| \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\beta u}{\partial x^\beta} \right|^2 dx, \qquad \|h\|^2_{G,l} = \sum_{|\beta|\leq l} \int_G \left| \frac{\partial^\beta u}{\partial x^\beta} \right|^2 dx$$

and define similarly $|||u|||_{\Gamma(t),l}$ with the aid of a partition of unity and local coordinates on $\Gamma(t)$. Then, if the smoothness assumptions on the matrices A_ν, B, and C are strengthened by order l, then under conditions I—III one establishes ...

the following inequality, generalizing (6), is established:

$$|||u|||^2_{G(t),\,l} + \int_0^t |||u|||^2_{\Gamma(\tau),\,l}\, d\tau \leq$$

$$\leq \text{const} \left\{ \int_0^t \left[|||f|||^2_{G(\tau),\,l} + |||g|||^2_{\Gamma(\tau),\,l} \right] d\tau + \|h\|^2_{G,\,l} \right\}, \qquad 0 \leq t \leq T. \tag{9}$$

Inequality (9) is proved by means of localization, application of Theorem 1 to the system obtained from (1)—(3) by differentiation, and use of the embedding theorems of N. Aronszajn—L. N. Slobodetskii (see, for example, (11)).

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Note: Figure translations are in progress. See original paper for figures.