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INFINITESIMAL BENDINGS OF SURFACES OF REVOLUTION OF POSITIVE CURVATURE WITH A CONICAL OR PARABOLIC POINT AT THE POLE

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It is known that infinitesimal bendings of surfaces of positive curvature are described by the generalized Cauchy–Riemann system ([1]). If there are isolated singular points on the surface, then we naturally arrive at a system of differential equations with singular coefficients ([2]). Under certain smallness conditions on the coefficients in ([2]), a theory has been constructed analogous to the well-known theory of I. N. Vekua, and two main boundary-value problems have been investigated. In the general case such systems have not been studied sufficiently.

In the present paper we consider the system corresponding to infinitesimal bendings of a surface of revolution of positive curvature with a conical or parabolic point at the pole. As boundary conditions, the variations of the normal curvature δk_s or of the geodesic torsion $\delta \tau_s$ are prescribed. The geometric results obtained here differ from the corresponding results for regular surfaces of strictly positive curvature.

1. The method by means of which the subsequent results are obtained is based on one way of reducing a system of first-order differential equations to a single second-order equation.

Lemma 1. *Suppose that in the system*

$$u_x - v_y = a(x,y)u + b(x,y)v, \qquad u_y + v_x = c(x,y)u + d(x,y)v, \tag{1}$$

$a, b, c, d \in C^1$, $u, v \in C^2$, *and the condition* $b_x + d_y = 0$ *is fulfilled. If* (u, v) *is a solution of system (1), then the function* $u(x, y)$ *satisfies the second-order equation*

$$\Delta u = (a + d)u_x + (c - b)u_y + [(bc - ad) + c_y]u. \tag{2}$$

Conversely, let $u(x, y)$ be a given solution of the second-order equation (2). Then there exists a function $v(x, y)$ (single-valued in a simply connected domain and multivalued in a multiply connected domain) such that the pair (u, v) satisfies system (1). For the effective construction of $v(x, y)$ it is necessary to integrate two exact differentials

$$dP = -d(x, y)\,dx + b(x, y)\,dy, \tag{3}$$

$$d\left(e^P v\right) = -e^P (u_y - cu)\,dx + e^P (u_x - au)\,dy. \tag{4}$$

Similar questions were considered in ([3]) for systems of a more general form, but with analytic right-hand sides and without an effective construction of the function $v(x, y)$. Various relations between systems (1) and second-order equations were investigated in the monograph of I. N. Vekua ([1]).

We apply the approach proposed in Lemma 1 to the investigation of boundary-value problems for systems with singular coefficients; however, it should be noted that it may prove useful in the study of many other questions.

Let $z = re^{i\theta} = x + iy$, $2\partial_{\bar z} = \partial_x + i\partial_y$, $w = u + iv$ be the unknown, and let $f(r)$ be a given real function. For the differential equation

$$\partial_{\bar z} w - \frac{1}{2} e^{i\theta} \frac{f(r)}{r - q}\, \overline{w} = 0, \qquad q < r < 1, \tag{5}$$

one is required to find a solution satisfying on the outer boundary the condition

$$\operatorname{Re} w\big|_{r=1} = u(1, \theta) = h(\theta). \tag{6}$$

On the inner boundary $r = q$ no condition is imposed other than boundedness of the solution.

Theorem 1. Let $f(r) \in C^1$ everywhere except at the point $r = q$, in a neighborhood of which

$$f'(r) = (r - q)^{\alpha - 1} f_0(r), \qquad 0 < \alpha < 1, \tag{7}$$

where $f_0(r)$ is a bounded function. It is assumed that $h(\theta)$ is expandable in an absolutely and uniformly convergent Fourier series and that

$$a(r) = f^2(r) - \frac{q}{r} f(r) + (r - q) f'(r) \geq 0, \qquad q < r \leq 1. \tag{8}$$

We consider solutions twice continuously differentiable inside the annulus and continuous everywhere except on the line $r = q$, where they are bounded.

If $f(q) > 1$, then the boundary-value problem (5), (6) is uniquely solvable for every $h(\theta)$.

If $f(q) < 0$, then the homogeneous problem has a solution of the form $w(z) = iC(r-q)^{|f(q)|}\Omega(r)$, where C is an arbitrary real constant and $\Omega(0) \neq 0, \infty$, and for solvability of the inhomogeneous problem it is necessary and sufficient that

$$\int_0^{2\pi} h(\theta)\, d\theta = 0.$$

In both cases the solution is continuous and vanishes on the singular line,

$$w(z) = O\big[(r-q)^{|f(q)|}\big]\,, \qquad r \to q.$$

The scheme of the proof of the theorem is as follows. If $w = u + iv$ is a solution of (5), then $u(x, y)$ will be a solution of the second-order equation

$$\Delta u - \frac{a(r)}{(r-q)^2} u = 0. \tag{9}$$

Under the conditions $a(r) \geq 0$, $a(q) > 0$, the solution of the boundary-value problem (9), (2) exists and is unique ([4,5]). The imaginary part $v(x, y)$ is found by formula (4), but since the integrand has a singularity, difficulties arise connected with the investigation of uniqueness and boundedness. To overcome them, the asymptotics of solutions of the boundary-value problem (9), (2) on the singular manifold is studied in detail.

Remark 1. The theorem is also valid for the problem

$$\operatorname{Im} w\big|_{r=1} \equiv v(1, \theta) = h(\theta), \tag{10}$$

if everywhere in the hypotheses and assertions of Theorem 1 $f(r)$ is replaced by $-f(r)$.

Remark 2. The theorem is also valid for $q = 0$, when the singular line turns into the singular point $r = 0$. It is necessary, however, to note that the inequality $f(q) > 1$ turns not into $f(0) > 1$, but into $f(0) > 0$. Thus, if in the case of a singular line the values $0 < f(q) < 1$ remain uninvestigated, then in the case of a singular point the problem is studied completely.

Let us note that the method used makes it possible to investigate a broader class of problems. First, this concerns the form of the domain. The outer bound-

can be not a circle but an arbitrary Lyapunov curve. Secondly, this concerns the coefficient of the equation, which need not have the form (5); it is sufficient to require that it satisfy the condition of Lemma 1. In this case it may have a singularity of high order. Thirdly, the theorem extends to inhomogeneous equations.

2. Let $\varphi = \varphi(\xi)$, $a \le \xi \le b$, be the equation of the meridian of a surface of revolution, and let θ, $0 \le \theta \le 2\pi$, be the angle formed by the plane of the meridian with some fixed plane passing through the axis of rotation. Everywhere in what follows it is assumed that $\varphi(a) = 0$, $\varphi(\xi) > 0$, and $\varphi''(\xi) < 0$, $a < \xi \le b$.

We pass from (ξ, θ) to the conjugate isothermal coordinate system (x, y), where $x = r\cos\theta$, $y = r\sin\theta$, and

$$r = \exp\left(-\int_\xi^b \sqrt{-\varphi''(\xi)/\varphi(\xi)}\, d\xi\right). \tag{11}$$

For the coefficients of the second quadratic form we shall have $L = N$, $M = 0$. For a given metric form of the surface, its various infinitesimal bendings can be characterized by the variations δL, δM, δN.

If we denote the Gaussian curvature by K and introduce the function

$$w = z^2 K^{1/4}(z)(\delta M + i\delta L), \tag{12}$$

then from the Gauss and Peterson–Codazzi equations we obtain

$$\partial_{\bar z} w + \frac{e^{i\theta}}{2}\frac{d}{dr}\ln\sqrt[4]{-\varphi^3\varphi''}\,\bar w = 0. \tag{13}$$

Specifying on the boundary parallel $\xi = b$ the variation of geodesic torsion $\delta\tau_s$ or of normal curvature δk_s, we arrive at the boundary condition (6) or (10), respectively.

Everywhere in what follows the admissible class of infinitesimal bendings is chosen so that in the above-mentioned conjugate isothermal coordinate system the variations of the coefficients of the second quadratic form are continuous everywhere, except for the pole, where they are bounded. Taking this into account and in accordance with the behavior at the pole of the Gaussian curvature $K(z)$, from (12) we obtain a definite requirement on the order of the solution $w(z)$. Depending on the nature of the singular point, the coefficient of equation (13) has a singularity of one type or another.

Pole–a conical point. Suppose that in a neighborhood of the pole $\xi = a$ the equation of the meridian has the form

$$\varphi(\xi) = \varphi'(a)(\xi - a) + (\xi - a)^m \varphi_0(\xi), \quad m > 1,$$

$$\varphi'(a) \neq 0, \infty, \quad \varphi_0(a) \neq 0, \infty, \quad \varphi_0(\xi) \in C^4. \tag{14}$$

The homeomorphism (11) maps the surface onto the circular annulus $q < r < 1$, where

$$q = \exp\left(-\int_a^b \sqrt{-\varphi''(\xi)/\varphi(\xi)}\, d\xi\right) > 0.$$

Taking (14) into account, it is not difficult to show that equation (13) assumes the form (5), where

$$f(r) = -\frac{3\varphi'\varphi'' + \varphi\varphi'''}{4\varphi''\sqrt{-\varphi\varphi''}}\frac{r-q}{r} \tag{15}$$

is continuous, $f(q) = -\frac{1}{2}(m+1)/2(m-1) < 0$, and $f'(r)$ has the form (7). Estimating the order of $K(z)$ near $r = q$, we arrive at the additional requirement

$$w(z) = O[(r-q)^\lambda], \quad \lambda = (m-3)/2(m-1), \quad r \to q. \tag{16}$$

Applying Theorem 1 and Remark 1, we obtain Theorems 2 and 3.

Theorem 2. *Let there be considered a surface of revolution of positive curvature whose boundary is a parallel and whose pole is a conical point. Suppose that the equation of the meridian near $\xi = a$ has the form (14) and is such that the function (15) satisfies condition (8). Then*

the surface admits a one-parameter family of infinitesimal bendings having a prescribed variation of the geodesic curvature of the boundary $\delta\tau_s$, if and only if the necessary and sufficient solvability condition is satisfied

$$\int_0^{2\pi} \delta\tau_s\, d\theta = 0.$$

Theorem 3. Let a variation of the normal curvature of the boundary δk_s be given, and let the equation of the meridian have the form (14) with $1 < m < 3$, and be such that the function (15) satisfies the condition

$$b(r) \equiv f^2(r) + \frac{q}{r}f(r) - (r-q)f'(r) \geq 0, \qquad q < r \leq 1. \tag{17}$$

Then the surface admits one and only one infinitesimal bending.

Let us note one of the geometric realizations of conditions (8) and (17). Since under the assumptions of Theorems 2 and 3 $a(q) = (3m-1)(m+1)/4(m-1)^2 > 0$, $b(q) = (m+1)(3-m)/4(m-1)^2 > 0$, the inequalities (8) and (17) are valid in some neighborhood of the point $r = q$. Therefore Theorems 2 and 3 are applicable to a surface that is a small neighborhood of a conical point.

Pole–a parabolic point. In a neighborhood of the pole $\xi = a$ ($\varphi = 0$) let us pass from $\varphi = \varphi(\xi)$ to the inverse function $\xi = \omega(\varphi)$. Suppose that

$$\omega(\varphi) = \omega(0) + \varphi^m \omega_0(\varphi), \quad m > 2, \qquad \omega_0(0) \neq 0, \infty, \qquad \omega_0(\varphi) \in C^4. \quad (18)$$

In this case the homeomorphism (11) maps the surface onto the disk $0 < r < 1$; $f(r)$ takes the form (15) with $q = 0$, moreover $f(0) = (m-2)/2\sqrt{m-1} = \nu$, and instead of (16) the condition $w(z) = O(r^{\nu+2})$, $r \to 0$, is required.

Using Remark 2, we arrive at the following conclusions.

Theorem 4. Let there be considered a surface of revolution of positive curvature, whose boundary is a parallel and whose pole is a parabolic point. Suppose that in a neighborhood of the pole the equation of the meridian has the form (18). Let the condition

$$a(r) \equiv f^2(r) + rf'(r) \geq 0, \qquad 0 < r \leq 1,$$

be satisfied, where $f(r)$ is given by formula (15) with $q = 0$. Then the surface under consideration admits one and only one infinitesimal bending having a prescribed variation of the geodesic curvature $\delta\tau_s$, if and only if $p_m = 2k_m + 1$ ($p_m \geq 5,\ p_m \to \infty$ as $m \to \infty$) necessary and sufficient conditions are fulfilled:

$$\int_0^{2\pi} \delta\tau_s \, d\theta = 0, \qquad \int_0^{2\pi} \delta\tau_s \cos k\theta \, d\theta = 0, \qquad \int_0^{2\pi} \delta\tau_s \sin k\theta \, d\theta = 0, \qquad k = 1, \dots, k_m, \quad (19)$$

where the integer k_m is determined by the inequalities

$$2\sqrt{1+\nu} - 1 \leq k_m < 2\sqrt{1+\nu}.$$

Similarly, if the condition

$$b(r) \equiv f^2(r) - rf'(r) \geq 0, \qquad 0 < r \leq 1,$$

is satisfied, then the surface admits one and only one infinitesimal bending having a prescribed variation of the normal curvature of the boundary δk_s, if and only if δk_s satisfies p_m conditions of the form (19).

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Note: Figure translations are in progress. See original paper for figures.