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Abstract

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ON THE STATISTICAL CHARACTERIZATION OF SYMMETRIC STABLE DISTRIBUTION LAWS

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Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be jointly independent random variables having one and the same distribution $F(x)$. A linear type of distributions is the set $\tilde{F}$ of distributions of the form $\{F(ax+b), a > 0, -\infty < b < \infty\}$, where $F(x)$ is the fixed distribution function generating the given type. We introduce classes $\mathfrak{A}_n,\ n \geqslant 1$, of types of distributions, defining them by the following conditions:

A. $\mathfrak{A}_m \cap \mathfrak{A}_n = \emptyset,\ m < n.$

B. $\tilde{F} \in \bigcup_{i=1}^{n} \mathfrak{A}_i$ if and only if, for every $\tilde{G} \neq \tilde{F}$, there exists a continuous bounded function $f(x_1, x_2, \dots, x_n)$ with the properties

$$f(ax_1 + b, ax_2 + b, \dots, ax_n + b) = f(x_1, x_2, \dots, x_n), \quad a > 0,\ -\infty < b < \infty;$$

$$M_{\tilde{F}} f(\xi_1, \xi_2, \dots, \xi_n) \neq M_{\tilde{G}} f(\xi_1, \xi_2, \dots, \xi_n).$$

In the author' s paper ([1]) it was shown that, for any $n \geqslant 1$, the set $\mathfrak{A}_n \cup \mathfrak{A}_{n+1} \cup \dots$ is nonempty. Moreover, an analogous assertion is valid for additive types of distributions. In the literature there are a number of results that can be interpreted as assigning concrete types of distributions to certain classes. Thus, the well-known result of A. A. Zinger ([2]) consists in the fact that the normal type of distributions belongs to $\mathfrak{A}_4 \cup \mathfrak{A}_5 \cup \mathfrak{A}_6$. In a recent article by Yu. V. Prokhorov ([3]) this result is extended to other symmetric laws.

In the present note the following assertion is proved.

Theorem. *The linear type generated by a symmetric stable distribution law belongs to* $\mathfrak{A}_3 \cup \mathfrak{A}_4 \cup \mathfrak{A}_5$.

In particular, A. A. Zinger' s theorem is refined as follows: the normal law is characterized by the totality of the distributions of the random variables

$$\eta_\alpha = \frac{\alpha\xi_1 + (1-\alpha)\xi_2 - \xi_3}{|\xi_4 - \xi_5|}, \quad 0 \leqslant \alpha \leqslant 1. \tag{1}$$

Proof of the theorem. Let $f(t) = Me^{it\xi_1}$. It is known ([4]) that

$$f(t) = \exp\{-|t|^\gamma\}, \tag{2}$$

where $0 < \gamma \leqslant 2$. (Since only statistics invariant with respect to shift and stretching will be considered, one may choose any distribution belonging to the given type.)

For brevity of notation, introduce the notation $\mathcal{L}[\varphi]$ for the distribution of any vector statistic φ, and the symbol $\Rightarrow$, understanding by the expression $\mathcal{L}[\varphi] \Rightarrow \mathcal{L}[\psi]$ that from the distribution of the statistic φ the distribution of ψ is uniquely recovered.

First of all,

$$\{\mathcal{L}[\operatorname{sign}\eta_\alpha,\ |\eta_\alpha|]\}_{0\leqslant\alpha\leqslant1} \Longleftrightarrow \{\mathcal{L}[\eta_\alpha]\}_{0\leqslant\alpha\leqslant1}. \tag{3}$$

Since in the present case the distribution of η_α is symmetric,

$$\mathcal{L}[|\eta_\alpha|] \Rightarrow \mathcal{L}[\operatorname{sign}\eta_\alpha,\ |\eta_\alpha|], \tag{4}$$

and therefore,

$$\{\mathcal{L}[|\eta_\alpha|]\}_{0\leq\alpha\leq1} \Rightarrow \{\mathcal{L}[\eta_\alpha]\}_{0\leq\alpha\leq1}. \tag{5}$$

Under condition (2), for $t = \operatorname{Re} t$

$$M\exp\{it[\alpha\xi_1 + (1-\alpha)\xi_2 - \xi_3]\} = f(\alpha t)f((1-\alpha)t)f(-t) =$$

$$= \exp\{-[\alpha^\gamma + (1-\alpha)^\gamma + 1]|t|^\gamma\}. \tag{6}$$

Introduce the random variable

$$\zeta_\alpha = |\alpha\xi_1 + (1-\alpha)\xi_1 - \xi_3|\,/[\alpha^\gamma + (1-\alpha)^\gamma + 1]^{1/\gamma}. \tag{7}$$

Then, by virtue of equality (6), the distribution of the random variables $\mu_\alpha = \zeta_\alpha|\xi_4 - \xi_5|^{-1}$ does not depend on α $(0 \leq \alpha \leq 1)$. On the other hand,

$$\{\mathcal{L}[\eta_\alpha]\}_{0\leq\alpha\leq1} \Longleftrightarrow \{\mathcal{L}[\mu_\alpha]\}_{0\leq\alpha\leq1}. \tag{8}$$

Further, we have

$$\ln\mu_\alpha = \ln\zeta_\alpha - \ln|\xi_4 - \xi_5|. \tag{9}$$

Let us prove that if ξ is a random variable with a symmetric stable distribution, then $\ln|\xi|$ has a characteristic function analytic in some strip containing $\{\operatorname{Im} z = 0\}$. Indeed, when $\operatorname{Im} x = \operatorname{Im} y = 0$,

$$|M \exp\{i(x+iy)\ln|\xi|\}| \leq M|\xi|^{-y} < \infty \qquad (-y < \gamma). \tag{10}$$

Consequently, the characteristic function of the random variable $\ln \mu_\alpha$ is not equal to 0 in any interval. Then from formula (9) the following conclusion follows. If the μ_α have distributions compatible with the hypothesis of stability of ξ_1, i.e., distributions not depending on α, then ζ_α will also have a distribution not depending on α ($0 \leq \alpha \leq 1$). Owing to symmetry, the distribution of the random variable $[\alpha\xi_1 + (1-\alpha)\xi_2 - \xi_3][\alpha^\gamma + (1-\alpha)^\gamma + 1]^{-1/\gamma}$ also does not depend on α and, in particular, is equal to the distribution corresponding to the case $\alpha = 1$. Thus we arrive at the functional equation

$$f(\alpha t) f((1-\alpha)t)\overline{f(t)} = \left| f\left(\left[\frac{\alpha^\gamma + (1-\alpha)^\gamma + 1}{2}\right]^{1/\gamma} t\right)\right|^2. \tag{11}$$

Denote

$$f(t) = e^{a(t) + ib(t)}, \tag{12}$$

where the newly introduced functions are real for $\operatorname{Im} t = 0$. Formula (11) is equivalent to two equalities

$$a(\alpha t) + a((1-\alpha)t) + a(t) = 2a\left(\left[\frac{\alpha^\gamma + (1-\alpha)^\gamma + 1}{2}\right]^{1/\gamma} t\right), \qquad a(0) = 0; \tag{13}$$

$$b(\alpha t) + b((1-\alpha)t) - b(t) = 0, \qquad b(0) = 0, \tag{14}$$

which hold in the interval $|t| < t_0 \leq \infty$, where $f(t)$ does not vanish.

From formula (13) it follows that the function $a(t)$ has growth of at most a power order. (For example, when

$$\left[\frac{\alpha^\gamma + (1-\alpha)^\gamma + 1}{2}\right]^{1/\gamma} = \rho_\alpha < 1,$$

taking into account that $a(t) \leq 0$, we have the inequality $|a(t)| \leq 2\max_{0<u<\rho_\alpha t} |a(u)|$.) Hence it follows that equations (13) and (14) are satisfied for all $t < 0$ and that the Laplace integral

$$\varphi(s) = \int_0^\infty e^{-st} a(t)\,dt \tag{15}$$

is analytic in the domain $\operatorname{Re} s > 0$. Equality (13) leads to the formula

$$\frac{1}{\alpha}\varphi\left(\frac{s}{\alpha}\right) + \frac{1}{1-\alpha}\varphi\left(\frac{s}{1-\alpha}\right) + \varphi(s) = \frac{2}{\rho_\alpha}\varphi\left(\frac{s}{\rho_\alpha}\right). \tag{16}$$

Expanding the terms of formula (16), except the first, in a Maclaurin series in α and making the obvious cancellations, we arrive at the conclusion that there exists a finite limit of the expression $\alpha^{-\gamma-1}\varphi(s/\alpha)$ as $\alpha \to 0$, and the differential equation is satisfied

$$s\varphi'(s) + \varphi(s) - cs^{\gamma-1} = 0. \tag{17}$$

Equations of this kind admit only solutions of the form

$$\varphi(s) = c_1 s^{-\gamma-1} + c_2 s^{-1}. \tag{18}$$

Substituting (18) into equation (16), for example for $\alpha = 1/2$, we find that $c_2 = 0$, and, therefore, by virtue of (15), $a(t) = ct^\gamma$, $t > 0$. In a similar way, from the functional equation (14) we find that $b(t) = t \cdot \text{const}$. The theorem is proved.

Remark. In the proof, the distribution of the statistic η_α was used not for all, but only for several values of α, namely: for $\alpha = 1$, at an infinitely close point $\alpha \to 1$, $\alpha = 1/2$, and also (for proving symmetry) at the point $\alpha = 0$ and at a point infinitely close to it. Therefore, from general analytic considerations there follows the possibility of generalizing our result in the direction indicated by Yu. V. Linnik and A. A. Zinger [5]; however, the author has not carried out the corresponding calculations.

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