ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF ONE-DIMENSIONAL SINGULAR DIFFERENTIAL OPERATORS

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.941.92

MATHEMATICS

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ASYMPTOTICS OF EIGENVALUES AND EIGENFUNCTIONS OF ONE-DIMENSIONAL SINGULAR DIFFERENTIAL OPERATORS

(Presented by Academician I. M. Vinogradov on 29 X 1965)

I. Consider the system

$$y' = \mu A(x) y, \tag{1}$$

where $A(x)$ is a square matrix of order n, y is a vector with n components, and μ is a parameter. Everywhere in what follows it is assumed that, for $x \geq 0$, the matrix $A''(x)$ is continuous (the prime denotes differentiation with respect to x; the matrix $A(x)$ is complex-valued). In this part of the article we study the asymptotics of solutions of system (1) as $x \to +\infty$.

We describe the form of the matrix $A(x)$.

1°. $A(x) = q(x)Q(x)B(x)Q^{-1}(x)$, where $q(x) \neq 0$ for $x \geq 0$ and Q is a diagonal matrix; $(Q(x))_{ii} = q^{\alpha_i}(x)$; α_i are complex numbers.

2°. The matrix $B(+\infty)$ exists, is finite, nondegenerate, and has distinct eigenvalues η_j.

3°.

$$\lim_{x\to+\infty} \left(|q'(x)| \, |\bar{q}(x)|^{-2} + |\bar{q}(x)|^{-1} \, \|B'(x)\| \right) = 0,$$

where

$$\|B(x)\| = \max_{i,j} |b_{ij}(x)|.$$

4°. The function

$$F(x) = |q'' q^{-3}| + \left| q'' q^{-2} + \left(|q|^{-1} + |q' q^{-2}| \right) \|B'\| + |q|^{-1} \left(\|B'\|^2 + \|B''\| \right) \right|$$

is summable on the interval $[0, +\infty)$.

Denote by $\lambda_j(x)$ the eigenvalues of the matrix $A(x)$, by $e_j(x), e_j^*(x)$ the right and left eigenvectors of this matrix corresponding to the eigenvalue $\lambda_j(x)$, and by $T(x)$ the matrix whose j-th column is $e_j(x)$, $1 \leq j \leq n$. Put

$$\lambda_j^{(1)}(x) = -(T^{-1}(x)T'(x))_{jj} \tag{2}$$

and introduce the diagonal matrices $\Lambda(x), \Lambda_1(x)$:

$$(\Lambda)_{jj} = \lambda_j(x), \qquad (\Lambda_1)_{jj} = \lambda_j^{(1)}(x). \tag{3}$$

All these notations, by virtue of 2°, have meaning for sufficiently large x.

Theorem 1. Suppose that conditions 1°–4° are satisfied,

$$\int_0^\infty |q(x)|\,dx = \infty, \tag{4}$$

and, for sufficiently large x, $i \neq j$,

$$\left|\mathrm{Re}(\lambda_i(x) - \lambda_j(x))\right| \geq C|q(x)|, \qquad C > 0. \tag{5}$$

Then for every $\mu_0 > 0$ there exists $x(\mu_0) < \infty$ such that, for $x \geq x(\mu_0)$, $\mu \geq \mu_0$, there exists a fundamental matrix $Y(x, \mu)$ of system (1) having the form

$$Y = T(x)(E + \mu^{-1}U(x,\mu)) \exp\left(\int_{x^0}^x (\mu\Lambda(t) + \Lambda_1(t))\,dt\right). \tag{6}$$

$$\|U(x,\mu)\| \leq u(x), \qquad \lim_{x\to+\infty} u(x) = 0. \tag{7}$$

Let

$$\lim_{x\to+\infty} |q(x)| = +\infty, \qquad \lim_{x\to+\infty} \arg q(x) = \varphi_0. \tag{8}$$

Denote

$$\eta_j' = \eta_j e^{i\varphi_0} \tag{9}$$

where η_j are the eigenvalues of the matrix $B(+\infty)$, and m^-, m^+ are the numbers of points η_j' lying respectively in the left and right half-planes, and introduce the operator

$$l = d/dx - A(x).$$

From Theorem 1 it follows:

Theorem 2. *Suppose that the conditions of Theorem 1 and condition (8) are satisfied and*

$$\operatorname{Re}\eta_j' \neq 0, \quad 1 \leq j \leq n; \qquad \operatorname{Re}(\eta_i' - \eta_j') \neq 0, \quad i \neq j. \tag{10}$$

Then, for any complex μ, the maximal number of linearly independent solutions of the system

$$ly = \mu y, \tag{11}$$

belonging to $L_2[0, +\infty)$, is equal to m^-.

Denote by D the collection of all vectors $y \in L_2[0, +\infty)$, all components of which are absolutely continuous on every finite interval $[0, a]$, $a > 0$, and such that $ly \in L_2[0, +\infty)$. By D_Ω we denote the collection of all vectors $y \in D$ such that

$$\Omega y(0) = 0, \tag{12}$$

where Ω is a constant matrix of rank m^-. By L_Ω we denote the operator in $L_2[0, +\infty)$ with domain of definition D_Ω, and $L_\Omega y = ly$ for $y \in D_\Omega$.

Theorem 3. *If the conditions of Theorem 2 are satisfied, then the spectrum of the operator L_Ω is purely discrete and has no finite limit points.*

Let us discuss the conditions of Theorem 1. If one requires that the matrix $A(x)$ be nondegenerate in a certain sense (see, for example, (1)) and that the $\lambda_j(x)$ have the same order of growth as $x \to +\infty$, then $A(x)$ has the form $1^\circ, 2^\circ$. In particular, for the scalar equation $y^{(n)} + \mu^n p(x)y = 0$ these conditions are automatically satisfied. If the characteristic roots of the scalar equation

$$\sum_{k=0}^{n} q_k(x) y^{(k)} = 0, \qquad q_n(x) \equiv 1,$$

have the same order of growth as $x \to +\infty$, then the matrix $A(x)$ has the form 1°, and $B(+\infty)$ exists and is finite; moreover $a_j = j - 1$. Therefore the main results of ([3]) are contained in Theorem 1. Theorem 1 also remains valid in the case where condition 2° is replaced by

$2^{0'}$. The matrix $B(+\infty)$ has no multiple roots.

Condition 3° is a restriction on the rate of decrease of $q(x)$ as $x \to +\infty$, and condition 4° is a restriction on the regularity of the behavior of $q(x)$ and $B(x)$ as $x \to +\infty$. For example, if $\|B^{(j)}(x)\| = O(x^{-j})$, $x \to +\infty$, $j = 1, 2$, and $q(x) = x^\alpha$, $\operatorname{Re}\alpha > -1$, then conditions 3°, 4° are satisfied. These conditions are also satisfied if $q(x)$ grows like $(\ln x)^\beta$ or like $\exp(Cx^\gamma)$, $\gamma > 0$, $\operatorname{Re} C > 0$, with arbitrary β, and this asymptotic expression can be differentiated twice. Condition (4) is satisfied in these examples. Finally, condition (5) can be weakened. In particular, if $A(x)$ is real, then it is sufficient to require (5) only for those pairs $\lambda_i(x), \lambda_j(x)$ which are not complex conjugates. G. Birkhoff ([2]) investigated the case where $\mu = 1$, $Q(x) = E$, $q(x) = x^r$, where $r \geq 0$ is an integer and $B(+\infty)$ has distinct eigenvalues; these results are contained in Theorem 1.

II. Consider the eigenvalue problem on the whole axis $(-\infty, +\infty)$ for system (1). A solution $y(x, \mu_0) \neq 0$ belonging to $L_2(-\infty, +\infty)$ is called an eigenfunction, and μ_0 is called an eigenvalue.

value. A point x_0 is called a turning point of system (1) if the matrix $A(x_0)$ has multiple eigenvalues. A turning point x_0 is called simple if $A(x_0)$ has a double eigenvalue λ_0, the remaining eigenvalues are simple, and $f_x(\lambda_0, x_0) \neq 0$, where $f(\lambda, x)$ is the characteristic polynomial of the matrix $A(x)$.

Introduce the conditions:

1. System (1) has on the real axis exactly two, and moreover simple, turning points $x_1 < x_2$, and $\lambda_m(x_j) = \lambda_{m+1}(x_j)$, $j = 1, 2$.
2. $\operatorname{Re}(\lambda_m(x) - \lambda_{m+1}(x)) \equiv 0$ for $x_1 \leq x \leq x_2$, $\operatorname{Re}\lambda_m(x) - \operatorname{Re}\lambda_{m+1}(x) < 0$ for $x > x_2,\ x < x_1$.
3. For all real x,
$$\operatorname{Re}(\lambda_j(x) - \lambda_{j+1}(x)) < 0, \quad j \neq m, n.$$

In particular, for the scalar equation $y'' - \mu^2 q(x) y = 0$, conditions 1-3 are satisfied if $q(x)$ is real for real x, has exactly two, and moreover simple, zeros $x_1 < x_2$ on the real axis, and $q(x) > 0$ for $x > x_2$.

Denote by $S(\rho, \alpha, \beta)$ the sector $|\mu| \geq \rho,\ \alpha < \arg\mu < \beta$.

Theorem 4. *Suppose the conditions of Theorem 1 and conditions 1-3 are satisfied on the real axis, $1 \leq m \leq n-1$, and the matrix $A(x)$ is regular on the interval $[x_1, x_2]$. Then there exist $\rho > 0,\ \alpha < 0 < \beta$ such that all eigenvalues of system (1) (for the problem on the whole axis) lying in $S(\rho, \alpha, \beta)$ have the form*

$$\mu_k = \xi_0^{-1}(2\pi k i + \xi_1) + O(k^{-1}), \quad k_0 \leq k < +\infty. \tag{13}$$

Here
$$\xi_0 = \frac{1}{2}\int_C (\lambda_{m+1} - \lambda_m)\, dt, \qquad \xi_1 = \frac{1}{2}\int_C (\lambda_{m+1}^{(1)} - \lambda_m^{(1)})\, dt, \tag{14}$$

where C is a closed contour in the complex x-plane which contains the interval $[x_1, x_2]$ in its interior, traverses it in the positive direction, and contains no other turning points of system (1) in its interior. The branches of the functions $\lambda_{m+1}, \lambda_m, \lambda_{m+1}^{(1)}, \lambda_m^{(1)}$ are chosen for $x \in C,\ x > x_2$.

We note that ξ_0 and ξ_1 are purely imaginary quantities and $\operatorname{Im} \xi_0 > 0$. In addition, the matrices $B(+\infty)$ and $B(-\infty)$ may be different, and $q(x)$ may have different orders of growth as $x \to \pm\infty$.

Corollary. *Let γ_0 be a smooth curve in the complex x-plane connecting two real points $x_3 < x_4$, and let*

$$\gamma = (-\infty, x_3] \cup \gamma_0 \cup [x_4, +\infty).$$

Suppose that on γ the conditions of Theorem 1, condition 1, and the following conditions hold:

$$2'.\quad \operatorname{Re}\left(\int_{x_1}^{x} (\lambda_{m+1} - \lambda_m)\, dt\right) \equiv 0, \quad \text{if } x \in \gamma \text{ and lies between } x_1 \text{ and } x_2;$$

$$\operatorname{Re}\left(\int_{x_1}^{x} (\lambda_{m+1} - \lambda_m)\, dt\right) > 0 \quad \text{for } x \in \gamma \text{ to the right of } x_2, \text{ and is less than zero for } x \in \gamma \text{ to the left of } x_1.$$

$$3'.\quad \operatorname{Re}\left(\int_{x_2}^{x} (\lambda_j - \lambda_{j+1})\, dt\right) < 0 \quad \text{for } j \neq m,\ x \in \gamma \text{ and lying to the right of } x_2,$$

and this function is positive for $x \in \gamma$ to the left of x_2.

Suppose, moreover, that the matrix $A(x)$ is regular for $x \in \overline{\gamma_0}$. Then all conclusions of Theorem 4 remain valid.

For the scalar equation $y'' - \mu^2 p(x) y = 0$, this question has been investigated more fully in ([4,5]).

Our methods also make it possible to find the asymptotics of the eigenfunctions as $\mu \to +\infty$ (in the case of Theorem 4) on the whole real axis, except for certain neighborhoods of the turning points.

Introduce the conditions:

1) System (1) has turning points on the real axis

$x_1 < x_2 < \cdots < x_{2l}$, all of them are simple, and $\lambda_{m-j+1}(x_j) = \lambda_{m+j}(x_{2l-j+1})$, $j = 1, 2, \ldots, l$.

2) $\operatorname{Re}(\lambda_{m-j+1}(x) - \lambda_{m+j}(x)) \equiv 0$ for $x \in [x_j, x_{2l-j+1}]$.

3) $\mathrm{Re}(\lambda_j(x) - \lambda_{j+1}(x)) < 0$ for all x, j, with the exception of those indicated in 2), and $\mathrm{Re}\,\lambda_{m+1}(x) > 0$, $\mathrm{Re}\,\lambda_m(x) < 0$ for $x \in [x_1, x_{2l}]$.

Theorem 5. *Suppose that the hypotheses of Theorem 1 and conditions 1)–3) are satisfied and that the matrix $A(x)$ is regular for $x \in [x_1, x_{2l}]$. Then there exist $\rho > 0$, $\alpha < 0 < \beta$ such that all eigenvalues of system (1) lying in $S(\rho, \alpha, \beta)$ have the form*

$$\mu_{kj} = \xi_{0j}^{-1}(2\pi k i + \xi_{1j}) + O(k^{-1}), \qquad j = 1, 2, \dots, l;\; k \geq k_0. \tag{15}$$

Here ξ_{0j}, ξ_{1j} have the form (14), where the integral is taken over the contour C_j enclosing the segment $[x_j, x_{2l-j+1}]$.

The quantities ξ_{0j} and ξ_{1j} are purely imaginary, and $\mathrm{Im}\,\xi_{0j} > 0$. The corollary from Theorem 4 remains valid. Similar formulas are obtained for the asymptotics of the eigenvalues for a problem on a half-line (say, on the half-line $x \geq x_0$, $x_1 < x_0 < x_2$, under the hypotheses of Theorem 4).

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Received
18 X 1965

REFERENCES

[1] L. R. Volevich, *Uspekhi Mat. Nauk*, 18, No. 3, 155 (1963).
[2] G. Birkhoff, *Trans. Am. Math. Soc.*, 10, 436 (1909).
[3] M. V. Fedoruk, *Dokl. Akad. Nauk*, 165, No. 4 (1965).
[4] Yu. N. Dnestrovsky, L. P. Kostomarov, *Zh. Vychisl. Mat. Mat. Fiz.*, 4, No. 2, 267 (1964).
[5] M. V. Fedoruk, *Mat. Sb.*, 68 (110), No. 1, 68 (1965).

Note: Figure translations are in progress. See original paper for figures.