ON AN INTEGRAL REPRESENTATION OF SMOOTH FUNCTIONS AND SOME FAMILIES OF FUNCTION SPACES

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.51

MATHEMATICS

A. D. DZHABRAILOV

ON AN INTEGRAL REPRESENTATION OF SMOOTH FUNCTIONS AND SOME FAMILIES OF FUNCTION SPACES

(Presented by Academician S. L. Sobolev on June 11, 1965)

1. We shall assume that e is any subset of the set of natural numbers $e_n = \{1, \dots, n\}$. If $h = (h_1, \dots, h_n)$ is a given vector, then let $h^e = (h_1^e, \dots, h_n^e)$, where $h_j^e = h_j$ for $j \in e$; $h_j^e = 0$ for $j \in e_n \setminus e$. Let $f(x)$ be a function defined in the Euclidean space E^n of points $x = (x_1, \dots, x_n)$; $\Delta_j^{k_j}(t_j) f(x)$ is the finite difference of order k_j with respect to the variable x_j with step t_j. Put

$$\Delta^{k^e}(t) f = \left[\prod_{j \in e} \Delta_j^{k_j}(t_j) \right] f.$$

For each subset $e \subseteq e_n$, $f^{(m^e)}(x)$ is the derivative of the function $f(x)$ of order m^e with respect to the variables x^e. The order of differentiation is arbitrary; for example:

$$f^{(m^e)}(x) = \frac{\partial^{m_1^e}}{\partial x_1^{m_1^e}} \cdots \frac{\partial^{m_n^e}}{\partial x_n^{m_n^e}} f(x).$$

Let

$$\int_0^{h^\sigma} (\cdot)\, dy^e = \int_0^{h_1^{\sigma_1}} dy_1^e \cdots \int_0^{h_n^{\sigma_n}} (\cdot)\, dy_n^e,$$

where

$$\int_0^{h_j^{\sigma_j}} dy_j^e = \int_0^{h_j^{\sigma_j}} dy_j \quad \text{for } j \in e,$$

and

$$\int_0^{h_j^{\sigma_j}} dy_j^e$$

is the identity operator for $j \in e_n \setminus e$.

Let Ω and e_n^* be any fixed subsets of the set e_n. Vectors $h = (h_1, \dots, h_n)$ and $\sigma = (\sigma_1, \dots, \sigma_n)$ with positive components are given, and a vector h is such that for $j \in e_n \setminus \Omega$, $h_j = h_0$. For all such vectors an integral representation has been obtained for sufficiently smooth functions f:

$$f^{(\nu)}(x) \equiv \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}} f(x) = \sum_{e \subset \Omega} \sum_{i \in e_n^0 \setminus \Omega} (-1)^{|e|+1} I_e^{(i)}(f). \tag{1}$$

Here $e_n^0 = \{0, 1, \dots, n\}$, $|e|$ is the number of elements of the set e,

$$I_e^{(0)}(f) \equiv I_e(f, e^*) =$$

$$= \int_0^h dv^e \int_0^{v^\sigma} dy^e \int_0^{v^\sigma - y} dt^{e^*} \int_0^{h^\sigma} \Delta^{k^{e^*}} \left(\frac{t}{k}\right) f^{(m^e)}(x + y)\, R_e\, dy^{e_n \setminus e},$$

where $e^* = e_n^* \cap e$, $m_i + k_i - \nu_i \geq 0$,

$$R_e = R_e(y, t^{e^*}, v^e, h^{e_n \setminus e});$$

$$I_e^{(i)}(f) = \begin{cases} I_{e \cup \{i\}}(f, e^* \cup \{i\}) & \text{for } i \in e_n^*, \\ I_{e \cup \{i\}}(f, e^*) & \text{for } i \in \overline{e_n^*}. \end{cases}$$

The integral kernels R_e are differentiable functions of their arguments and admit an estimate in terms of the components of the vector h.

The integral representation (1) for $e_n^* = e_n$, $\Omega = \emptyset$, i.e., when Ω is the empty set, coincides with the integral representation obtained by V. P. Il' in (^{13}).

2. Let $r = (r_1, \dots, r_n)$ be a vector with nonnegative components. By the **supporting vectors** of the vector r we shall mean the smallest subset e of the set e_n such that $r^e \equiv r$. We denote the support of this vector by e_r. Suppose that e_r^* is the set of those indices j from e_r for which, when $j \in e_r^*$, the number r_j is not an integer.

Let e be any subset of the set e_r, and let $e^* = e_r^* \cap e$. For each positive r_j put $r_j = \bar{r}_j + \alpha_j$, where $\bar{r}_j$ is the integer part of r_j, so that $0 \leq \alpha_j < 1$, and if $r_j = 0$, put $\bar{r}_j = 0$. Thus to each vector $r = (r_1, \dots, r_n)$ there corresponds a vector $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$.

Definition. Let e be such a subset of the set e_r that $e^* = \emptyset$, i.e., all r_j $(j \in e)$ are integers. We shall say that $f \in L_p^{(r^e)}(E^n)$ if the function $f(x)$ in E^n has a generalized derivative in the sense of S. L. Sobolev (^{1}), $f^{(r^e)}(x) \in L_p(E^n)$ $(p \geq 1)$. The norm in this space is defined as follows:

$$\|f, L_p^{(r^e)}(E^n)\| = \|f^{(r^e)}, L_p(E^n)\| = \left(\int_{E^n} |f^{(r^e)}(x)|^p \, dE^n\right)^{1/p}.$$

Now let e be such a subset of the set e_r that it intersects the set e_r^*, i.e. the set e^* is nonempty. For such e we shall say that $f \in L_p^{(r^e)}(E^n)$, if $f \in L_p^{(\bar{r}^e)}(E^n)$ and the integral

$$\mathcal{J}_e(f) = \left(\int_0^\infty \cdots \int_0^\infty \|\Delta^{\omega e^*}(t) f^{(\bar{r}^e)}, L_p(E^n)\|^p \frac{dt^{e^*}}{\prod_{j \in e^*} t_j^{1+p\alpha_j}} \right)^{1/p}$$

is finite, where $\omega = (1, \dots, 1)$ is the vector all of whose components are equal to one; and we define the norm in this space by

$$\|f, L_p^{(r^e)}(E^n)\| = \|f, L_p^{(\bar{r}^e)}(E^n)\| + \mathcal{J}_e(f).$$

Let Ω be any fixed subset of the set e_r, and let the set e be a subset of the set Ω.

Definition. For $i \in e_r^*$ we shall say that $f \in L_p^{(r^e, r_i)}(E^n)$, if the generalized derivative

$$f^{(r_i)}(x) \equiv \frac{\partial^{r_i}}{\partial x_i^{r_i}} f(x) \in L_p^{(r^e)}(E^n)$$

and we define the norm as follows:

$$\|f, L_p^{(r^e, r_i)}(E^n)\| = \|f^{(r_i)}, L_p^{r^e}(E^n)\|.$$

For $i \in e_r^*$ we shall say that $f \in L_p^{(r^e, r_i)}(E^n)$, if $f \in L_p^{(r^e, \bar{r}_i)}(E^n)$ and the integral

$$\mathcal{J}_e^{(i)}(f) = \left\{ \int_0^\infty [\mathcal{J}_e(\Delta_i(t_i) f^{(\bar{r}_i)})]^p t_i^{-(1+p\alpha_i)}\, dt_i \right\}^{1/p}$$

is finite, and we define the norm by

$$\|f, L_p^{(r^e, r_i)}(E^n)\| = \|f, L_p^{(r^e, \bar{r}_i)}(E^n)\| + \mathcal{J}_e^{(i)}(f).$$

Basic definition. We shall say that $f \in W_p^{(r)}(\Omega, E^n)$, if for every $e \subset \Omega$ and every $i \in e_r^0 \setminus \Omega$, $f \in L_p^{(r^e, r_i)}(E^n)$, where $e_r^0 =$

$= e_r \cup \{0\},\ r_0 = 0$, and define the norm in this space as follows:

$$\|f, W_p^{(r)}(\Omega, E^n)\| = \sum_{e \subset \Omega} \sum_{i \in e_r^0 \setminus \Omega} \|f, L_p^{(r^e, r_i)}(E^n)\|,$$

where

$$L_p^{(\emptyset,0)}(E^n) \equiv L_p(E^n), \qquad L_p^{(r^e,0)}(E^n) \equiv L_p^{(r^e)}(E^n).$$

The space $W_p^{(r)}(\Omega, E^n)$, for $\Omega = \emptyset$ and $e_r = e_n$, coincides with the generalized space of S. L. Sobolev $W_p^{(r_1,\dots,r_n)}(E^n)$ (see (^{1}, ^{5}, ^{7})), and for $\Omega = e_r$ it coincides with the space $S_p^{(r)}W(E^n)$, which for integral r_i ($i \in e_r$) was defined by S. M. Nikol' skii (^{4}).

Definition. By the space $\mathcal{W}_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of smooth finite functions in the norm $\|f, W_p^{(r)}(\Omega, E^n)\|$.

Theorem 1. *The spaces* $\mathcal{W}_p^{(r)}(\Omega, E^n)$ *and* $W_p^{(r)}(\Omega, E^n)$ *coincide for* $1 < p < \infty$.

3. Let $r = (r_1, \dots, r_n)$ be a vector with nonnegative components r_i ($i = 1, \dots, n$), whose support is e_r. For each positive r_j put $r_j = \bar{r}_j + \beta_j$, where $\bar{r}_j$ is the greatest integer less than r_j, so that $0 < \beta_j \le 1$, and when $r_j = 0$ put $\bar{r}_j = 0$; consequently, to each vector $r = (r_1, \dots, r_n)$ there corresponds the vector $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$. Introduce the norms

$$\|f, \mathcal{L}_p^{(r^e)}(E^n)\| = \left(\int_0^\infty \cdots \int_0^\infty \|\Delta^{2\omega^e}(t) f^{(\bar{r}^e)}, L_p(E^n)\|^p \frac{dt^e}{\prod_{j\in e} t_j^{1+p\beta_j}} \right)^{1/p},$$

where e is any subset of the set e_r;

$$\|f, \mathcal{L}_p^{(r^e, r_i)}(E^n)\| = \left(\int_0^\infty \|\Delta_i^2(t_i) f^{(\bar{r}_i)}, \mathcal{L}_p^{(r^e)}(E^n)\|^p \frac{dt_i}{t_i^{1+p\beta_i}} \right)^{1/p},$$

where e is any subset of the set $\Omega \subset e_r$ and $i \in e_r \setminus \Omega$;

$$\|f, B_p^{(r)}(\Omega, E^n)\| = \sum_{e\subset\Omega} \sum_{i\in e_r^0\setminus\Omega} \|f, \mathcal{L}_p^{(r^e, r_i)}(E^n)\|,$$

where

$$e_r^0 = e_r \cup \{0\}, \qquad \mathcal{L}_p^{(\emptyset,0)}(E^n) \equiv L_p(E^n), \qquad r_0 = 0.$$

Definition. By the space $B_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of sufficiently smooth finite functions in the norm $\|f, B_p^{(r)}(\Omega, E^n)\|$. The space $B_p^{(r)}(\Omega, E^n)$, for $\Omega = \emptyset$ and $e_r = e_n$, coincides with the known space $B_p^{(r_1,\dots,r_n)}(E^n)$, defined and studied by O. V. Besov (^{6}) (see also (^{7}, ^{8}, ^{10}), etc.), and for $\Omega = e_r$ it coincides with the known space $S_p^{(r)}B(E^n)$, defined by the author (^{12}).

4. Let, for any subset e of the set e_r,

$$M^{(r^e)}(f) = \sup_{t^e} \left\| \prod_{j\in e} t_j^{-\beta_j} \Delta^{2\omega^e}(t) f^{(\bar{r}^e)},\ L_p(E^n) \right\|.$$

For any subset e of the set $\Omega \subset e_r$ and any $i \in e_r \setminus \Omega$ put

$$M^{(r^e, r_i)}(f) = \sup_{t_i} t_i^{-\beta_i} M^{(r^e)}(\Delta_i^2(t_i) f^{(\bar{r}_i)}).$$

Definition. By the space $H_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of sufficiently smooth finite functions in the norm

$$\|f, H_p^{(r)}(\Omega, E^n)\| = \sum_{e\subset\Omega} \sum_{i\in e_r^0\setminus\Omega} M^{(r^e, r_i)}(f),$$

where

$$e_r^0 = e_r \cup \{0\}, \qquad M^{(\emptyset,0)}(f) = \|f, L_p(E^n)\|.$$

An analogous remark is also made for $H_p^{(r)}(\Omega, E^n)$ (see (^{2,3,7,11}).

5. Let the vector $r = (r_1, \dots, r_n)$ be such that $e_r = e_n$, i.e., the components of this vector are positive numbers. Further, let $1 < p \le q \le \infty$, let Ω be any fixed subset of the set $e_n = \{1, \dots, n\}$, let m be any natural number $\le n$ and $e_m = \{1, \dots, m\}$, and let $\nu = (\nu_1, \dots, \nu_n)$ be a vector with nonnegative integer components. Put

$$\varepsilon_j = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{r_j} - \frac{\nu_j}{r_j} > 0 \quad (j \in \Omega \cap e_m \equiv \Omega^*), \qquad \varepsilon_i = 1 - \frac{1}{pr_i} - \frac{\nu_i}{r_i} > 0$$

$$(i \in \Omega \setminus \Omega^*), \qquad \varepsilon_\Omega = 1 - \frac{1}{p}\sum_{j\in e_n\setminus\Omega} \frac{1}{r_j} - \sum_{j\in e_n\setminus\Omega} \frac{\nu_j}{r_j} + \frac{1}{q}\sum_{j\in e_m\setminus\Omega^*} \frac{1}{r_j} > 0.$$

Theorem 2. Let $f \in W_p^{(r)}(\Omega, E^n)$, and let the vector $\rho = (\rho_1, \dots, \rho_m)$ be such that its components satisfy the conditions:

a) for $j \in \Omega^*$, $0 < \rho_j < \varepsilon_j r_j$, if at least one of ρ_j and r_j is an integer; $0 < \rho_j \le \varepsilon_j r_j$, if ρ_j and r_j are simultaneously nonintegers;

b) for $j \in e_m \setminus \Omega^*$, $0 < \rho_j < \varepsilon_\Omega r_j$, if at least one of ρ_j and r_i $(i \in e_n \setminus \Omega)$ is an integer; $0 < \rho_j \le \varepsilon_\Omega r_j$, if ρ_j and r_i $(i \in e_n \setminus \Omega)$ are simultaneously nonintegers.

Then, for any fixed $x_{m+1}, \dots, x_n$, $f^{(\nu)} \in W_q^{(\rho)}(\Omega^*, E^m)$, and the inequality holds

$$\|f^{(\nu)}, W_q^{(\rho)}(\Omega^*, E^m)\| \le c\|f, W_p^{(r)}(\Omega, E^n)\|,$$

where c is a constant independent of f.

For brevity we shall denote the assertion of this theorem as follows:

$$W_p^{(r)}(\Omega, E^n) \to W_q^{(\rho)}(\Omega^*, E^m).$$

Theorem 3.

$$B_p^{(r)}(\Omega, E^n) \to B_q^{(\rho)}(\Omega^*, E^m),$$

where the components of the vector $\rho = (\rho_1, \dots, \rho_m)$ satisfy the conditions: $0 < \rho_j \le \varepsilon_j r_j$ for $j \in \Omega^*$; $0 < \rho_j \le \varepsilon_\Omega r_j$ for $j \in e_m \setminus \Omega^*$.

Theorem 4. Under the conditions of Theorem 3,

$$H_p^{(r)}(\Omega, E^n) \to H_q^{(\rho)}(\Omega^*, E^m).$$

Theorem 5.

$$W_p^{(r)}(\Omega, E^n) \to B_q^{(\rho)}(\Omega^*, E^m),$$

where the components of the vector $\rho = (\rho_1, \dots, \rho_m)$ satisfy the conditions:

a) for $j \in \Omega^*$, $0 < \rho_j < \varepsilon_j r_j$, if r_j is an integer; $0 < \rho_j \le \varepsilon_j r_j$, if r_j is noninteger;

b) for $j \in e_m \setminus \Omega^*$, $0 < \rho_j < \varepsilon_\Omega r_j$, if at least one of the r_i $(i \in e_n \setminus \Omega)$ is an integer; $0 < \rho_j \le \varepsilon_\Omega r_j$, if all r_i $(i \in e_n \setminus \Omega)$ are nonintegers.

Theorem 6. Under the conditions of Theorem 3,

$$B_p^{(r)}(\Omega, E^n) \to H_q^{(\rho)}(\Omega^*, E^m).$$

I express my sincere gratitude to Academician S. L. Sobolev and Professor L. D. Kudryavtsev for their attention to this work.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
9 IV 1965

CITED LITERATURE

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[2] S. M. Nikol' skii, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **38** (1951).
[3] L. D. Kudryavtsev, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **55** (1959).
[4] S. M. Nikol' skii, Matem. sborn., **61** (103), 2 (1963).
[5] L. N. Slobodetskii, Uch. zap. LGU, fiz.-matem. fak., **197** (1958).
[6] O. V. Besov, Tr. Matem. inst. V. A. Steklova AN SSSR, **60** (1961).
[7] V. P. Il' in, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **66** (1962).
[8] S. V. Uspenskii, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **60** (1961).
[9] S. M. Nikol' skii, Sibirsk. matem. zhurn., **4**, No. 6 (1963).
[10] P. I. Lizorkin, Matem. sborn., **60** (102), 3 (1963).
[11] Ya. S. Burenkov, Izv. AN SSSR, ser. matem., **22** (1958).
[12] A. D. Dzhabrailov, DAN, **159**, No. 2 (1964).
[13] V. P. Il' in, V. A. Solonnikov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **66** (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.