EMBEDDING THEOREMS FOR FUNCTIONS DEFINED ON UNBOUNDED DOMAINS, AND THEIR APPLICATION TO BOUNDARY-VALUE PROBLEMS FOR THE POLYHARMONIC EQUATION

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Abstract

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EMBEDDING THEOREMS FOR FUNCTIONS DEFINED ON UNBOUNDED DOMAINS, AND THEIR APPLICATION TO BOUNDARY-VALUE PROBLEMS FOR THE POLYHARMONIC EQUATION

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Let E_n be the n-dimensional Euclidean space of points $x = (x_1, x_2, \dots, x_n)$; E_n^+ the upper half-space $x_n > 0$; E_m an m-dimensional hyperplane; $\rho = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$; $|x - y|$ the distance between the points x and y; Q_n the n-dimensional ball of radius one with center at the origin; α, r, s non-negative numbers; $\bar{r} = [r]$ the integer part of the number r; $1 < p < \infty$; $u^{(k)} = \partial^{|k|} u / \partial x_1^{k_1} \dots \partial x_n^{k_n}$, $k = (k_1, k_2, \dots, k_n)$ an integer vector, $k_j \geq 0$, $|k| = k_1 + k_2 + \dots + k_n$;

$$|u, L_p(E_n)| = \left\{ \int_{E_n} |u|^p \, dE_n \right\}^{1/p}.$$

We consider the class $L^r_{p,\alpha}(E_n)$ of functions $u(x)$, defined in E_n together with their generalized derivatives up to order $\bar{r}$ inclusive, for which the norm is finite

$$|u, L^r_{p,\alpha}(E_n)| = \sum_r \left| \frac{u^{(r)}(x)}{(1+\rho)^\alpha}, \, L_p(E_n) \right|,$$

if r is an integer, or

$$|u, L^r_{p,\alpha}(E_n)| = \sum_r \left\{ \int_{E_n} dE_n^x \frac{|u^{(\bar{r})}(x) - u^{(\bar{r})}(y)|^p}{|x-y|^{n+(r-\bar{r})p}} \frac{dE_n^y}{(1 + \rho(x) + \rho(y))^{\alpha p}} \right\}^{1/p},$$

if r is not an integer.

The norm in the class $W^r_{p,\alpha}(E_n)$ is defined as follows:

$$|u, W^r_{p,\alpha}(E_n)|^p = |u, L^r_{p,\alpha}(E_n)|^p + |u, L^r_p(Q_n)|^p.$$

The norms in the classes $L^r_{p,\alpha}(E^+_n)$, $W^r_{p,\alpha}(E^+_n)$ are defined analogously. Classes of functions in which finiteness of the norm of the function itself in $L_p(E_n)$ is not required were first considered by L. D. Kudryavtsev ([1,2]), which made it possible to solve the first boundary-value problem for an elliptic equation of second order in a new way, by extending the class of boundary values ([2–3]).

The norm in the fractional class $L^r_{p,\alpha}(E_n)$, r not an integer, may also be introduced in another way (as was done by S. M. Nikol' skii ([5]) for a weight function of a more general kind):

$$^*|u, L^r_{p,\alpha}(E_n)| = \sum_r \sum_{j=1}^n \left\{ \int_0^\infty \frac{dh}{h^{1+(r-\bar{r})p}} \,|\sigma \Delta_j u^{(\bar{r})}, L_p(F_n)|^p \right\}^{1/p};$$

here Δ_j is the first difference with step h in the j-th coordinate, σ is a weight function, but in the case of the weight considered by us these norms are equivalent,

i.e., the estimate holds

$$\left|u, L^r_{p,\alpha}(E_n)\right| \le c_1^* \left|u, L^r_{p,\alpha}(E_n)\right| \le c_2 \left|u, L^r_{p,\alpha}(E_n)\right|.$$

For the classes defined above, the following assertions are proved.

Theorem 1. If r, k are integers, $\alpha > n/p - 1$, $1 \le k \le r$, $r - k - (n-m)/p > 0$, then the embedding holds

$$W^r_{p,\alpha}(E^+_n) \to W^{r-k}_{p,\alpha+k-(n-m)/p}(E_m).$$

Theorem 2. If r is an integer and either a) $\alpha > n/p - 1$, $(n-m)/p \ge 1$, $r - k - (n-m)/p > 0$, or b) $\alpha \ge 0$, $(n-m)/p < 1$, $k = 0$, then the embedding is valid

$$W^r_{p,\alpha}(E^+_n) \to W^{r-k-(n-m)/p}_{p,\alpha+k}(E_m).$$

Theorem 3. If r is not an integer, $\alpha \ge 0$, $\bar{r} - k \ge 0$, then

$$W^r_{p,\alpha}(E_n) \to W^{\bar{r}-k}_{p,\alpha+n/p+k+(r-\bar{r})}(E_n).$$

Theorem 4. If r is not an integer, $r - k - (n-m)/p > 0$, $\alpha \ge 0$, then the embedding is valid

$$W^r_{p,\alpha}(E^+_n) \to W^{r-k-(n-m)/p}_{p,\alpha+n/p+k}(E_m),$$

where $k > 0$ when $(r-k-(n-m)/p)$ is an integer, and $k \geq 0$ when $(r-k-(n-m)/p)$ is fractional.

Examples have been constructed showing that these results are sharp in the sense that, when the exponent of the weight is decreased, the theorems cease to be true.

Consider the polyharmonic equation

$$\Delta^m u = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right)^m u = 0. \tag{1}$$

Let Ω be an unbounded domain of n-dimensional space with boundary Γ consisting of hyperplanes of various dimensions Γ_i.

Equation (1) is the Euler equation for the integral

$$D^m(u) = \int_\Omega \cdots \int \sum_{\Sigma k_i = m} \frac{m!}{k_1! k_2! \cdots k_n!} \left(\frac{\partial^m u}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}\right)^2 d\Omega, \tag{2}$$

which is equivalent to the norm of the function u in the space $L^m_2(\Omega)$.

Let $u \in W^{2m}_2(\Omega)$. From Theorem 1 it follows that the derivatives of order k of the function u belong to $L_{2,\alpha+m-k-(n-s)/2}$ on hyperplanes of dimension s, where $m-k-(n-s)/2 > 0$, or $n-s < 2(m-k)$, whence it follows that

$$k \leq m - [(n-s)/2] - 1.$$

Let the boundary Γ of the domain Ω consist of hyperplanes $\Gamma_{n-1}, \dots, \Gamma_{n-2m+1}$ (some of which may be absent), and suppose that on each of these hyperplanes functions are given

$$\varphi^s_{k_1,\dots,k_n}, \qquad \text{where } 0 \leq \sum_i k_i \leq m - [(n-s)/2] - 1.$$

If there exists a function $v \in W^{2m}_2(\Omega)$ such that

$$\partial^{|k|} v / \partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n} \big|_{\Gamma_s} = \varphi^s_{k_1,\dots,k_n}, \tag{3}$$

then we say that this system of boundary values $\{\varphi^s_{k_1,\dots,k_n}\}$ is admissible.

If the system $\{\varphi^s_{k_1,\dots,k_n}\}$ is admissible, then from the embedding theorems we may assert that

$$\varphi^s_{k_1,\dots,k_n} \in L_{2,\alpha}^{m-|k|-(n-s)/2}(\Gamma_s), \quad s \neq n-1, \qquad \varphi^{n-1}_{k_1,\dots,k_n} \in L_2^{m-|k|-1/2}(\Gamma_{n-1}).$$

Theorem 5. *If the system of functions $\{\varphi^s_{k_1,\dots,k_n}\}$ is admissible, then there exists, and moreover a unique, function $u \in W_2^m(\Omega)$ satisfying (3) and giving a minimum to the integral (2) among all such functions. This function u is a generalized solution of equation (1), and in the class of functions $W_2^m(\Omega)$ satisfying (3) it is unique.*

If Ω is the upper half-space E_n^+, then, prescribing on E_{n-1} the system of functions φ_i $(i = 0, 1, 2, \dots, m-1)$, and considering the class $D^m\{\varphi_i\}$ of those functions u from $W_2^m(E_n^+)$ which have boundary values φ_i, i.e.

$$\partial^i u/\partial x_n^i\big|_{E_{n-1}} = \varphi_i,$$

and following the usual scheme ([4],[6]), we obtain:

Theorem 6. *In every nonempty class $D^m\{\varphi_i\}$ there exists a unique classical solution of equation (1).*

Necessary and sufficient conditions for the nonemptiness of the class were obtained in ([1],[5]).

In proving the embedding theorems, the integral representation for smooth functions ([7]) was used. Theorem 1, when $m = n-1$, and Theorem 2, case b), for $m = n-1$, were obtained earlier by L. D. Kudryavtsev ([1]).

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