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Abstract

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MATHEMATICS

Yu. A. ROZANOV

ON GAUSSIAN LINEAR FUNCTIONALS ON A COUNTABLY HILBERT SPACE

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Let U be a real separable countably Hilbert space, whose topology is given by means of a countable number of scalar products

$$(u, v)_1, \quad (u, v)_2, \dots, \tag{1}$$

$X = U'$ is the set of continuous linear functionals $x = \langle u, x\rangle$ on the space U; $\mathfrak{A}$ is the σ-algebra of Borel sets (with respect to the weak topology) of the space X.

A probability measure $P(dx)$ on the σ-algebra $\mathfrak{A}$ is called **Gaussian** if, for every $u \in U$, the random variable $u = \langle u, x\rangle$, considered as a function of $x \in X$, is Gaussian. This is equivalent to saying that the characteristic functional

$$\chi(u) = \int_X e^{i\langle u,x\rangle} P(dx)$$

of the measure $P(dx)$ has the form

$$\chi(u) = e^{ia(u) - \frac{1}{2}b(u,u)},$$

where

$$a(u) = \int_X \langle u, x\rangle P(dx), \tag{2}$$

$$b(u, v) = \int_X [\langle u, x\rangle - a(u)][\langle v, x\rangle - a(v)] P(dx). \tag{3}$$

Measures in linear topological spaces and arbitrary Gaussian measures have been the subject of numerous and relatively recent investigations. The principal results are contained partly already in the monograph ([1]), in survey articles ([2], [3]) (see also ([4])). The facts given below, concerning Gaussian measures in the space of linear functionals on a countably Hilbert space, are a simple consequence of these results.

Let $P(dx)$ be a Gaussian measure on the σ-algebra $\mathfrak{A}$ of the space $X = U'$, and let $a(u)$ and $b(u,v)$ be the functionals on the countably Hilbert space U defined by formulas (2) and (3); $a(u)$ is called the **mathematical expectation**, and $b(u,v)$ the **correlation functional**.

$a(u)$, $b(u,v)$ $(u, v \in U)$ are the mathematical expectation and correlation functional of some Gaussian measure if and only if they are representable in the form

$$a(u) = (u,a)_n; \qquad b(u,v) = (Bu,v)_n, \tag{4}$$

where $(u,v)_n$ is some scalar product from (1); a is an element of the Hilbert space $\bar{U} = U_n$ with this scalar product, obtained by completing U; B is a positive nuclear operator in $\bar{U}$.

Let us establish, for example, formula (4) for the correlation functional—

$b(u,v)$. As is known, this is a positive bilinear functional, which for some n is bounded not only on the sphere $(u,u)_n \leqslant 1$, but also on a set of the form $(Su,u)_n \leqslant 1$, where S is a positive nuclear operator in $\overline{U} = U_n$ (see ([2]), p. 414). This says that $b(u,v)$ is representable in the form (4), where B is a linear positive operator such that $Bu = 0$ when $Su = 0$ and the product BS^{-1} is bounded. If $u_1, u_2, ...$ is an orthonormal system of eigenvectors of the nuclear operator S, corresponding to positive eigenvalues $\lambda_1, \lambda_2, ...$, then $\|BS^{-1}u_k\| = \lambda_k^{-1}\|Bu_k\| \leq C$, whence it is seen that, together with the series $\sum \lambda_k$, the series $\sum \|Bu_k\|$ also converges. Consequently, B is a nuclear operator (see ([1]), p. 65).

Let us next consider two Gaussian measures $P(dx)$ and $P_1(dx)$. Like any Gaussian measures, they are either mutually absolutely continuous (that is, equivalent) or perpendicular. We indicate conditions for their equivalence and an expression for the corresponding density $p(x) = P_1(dx)/P(dx)$.

For the equivalence of $P(dx)$ and $P_1(dx)$ it is necessary and sufficient that, first, the corresponding mathematical expectations $a(u)$, $a_1(u)$ and correlation functionals $b(u,v)$, $b_1(u,v)$ have the form (4) for some common n; second, the corresponding nuclear operators B and B_1 in the Hilbert space $\overline{U} = U_n$ have the same null subspace $\overline{U}^0$, and the difference $I - C$, $C \equiv B^{-1/2}B_1B^{-1/2}$, be a Hilbert-Schmidt operator in the subspace $\overline{U}^+$ orthogonal to $\overline{U}^0$; third, the difference of the corresponding elements $a_1 - a$, defining by formula (4) the mathematical expectations $a(u)$ and $a_1(u)$, belong to the domain of definition of the operator $B^{-1/2}$.

This fact (cf. [5]) follows easily from the general condition for equivalence of Gaussian measures (see [3], p. 449), which in the case under consideration can be reformulated as follows: for a complete system in $\overline{U}^+$ of elements $v_1, v_2, \dots$ such that $(Bv_k, v_k)_n = 1$ and

$$(Bv_k, v_j)_n = 0 \quad \text{for } k \neq j; \quad k, j = 1, 2, \dots,$$

the series

$$\sum_{k,j} \left|(Bv_k, v_j)_n - (B_1 v_k, v_j)_n\right|^2$$

and

$$\sum_k (v_k, a_1 - a)^2$$

converge, with $a_1 - a \in \overline{U}^+$. If one puts

$$u_k = B^{1/2} v_k,$$

then $u_1, u_2, \dots$ will be a complete orthonormal system in $\overline{U}^+$. Moreover,

$$(Bv_k, v_j)_n - (B_1 v_k, v_j)_n = (I - Cu_k, u_j)_n$$

and the series

$$\sum_{k,j} (I - Cu_k, u_j)_n^2,$$

$$\sum_k (B^{-1/2} u_k, a_1 - a)_n^2$$

are convergent. This is equivalent to the fact that the difference $I - C$ is a Hilbert-Schmidt operator (see [1], p. 50) and that the element $a_1 - a$ belongs to the domain of definition of the operator $B^{-1/2}$.

In order to find an expression for the density $p(x) = P_1(dx)/P(dx)$ of the equivalent measures $P(dx)$ and $P_1(dx)$, we turn to the above-mentioned space $\overline{U} = U_n$. On it let us consider the new scalar product

$$(u, v) = (Bu, v)_n.$$

Let $\overline{\overline{U}}$ denote the corresponding completion of the space $\overline{U}$ with respect to this scalar product. To each element u of the Hilbert space $\overline{\overline{U}}$ there corresponds a certain measurable function $u = u(x)$ of x. This correspondence is such that $u(x) = \langle u, x \rangle$ for $u \in U$, and to a convergent sequence of elements $u_1, u_2, ...$ there corresponds a sequence of functions $u_1 = u_1(x), u_2 = u_2(x), ...$ converging in mean square. The operator $B^{-1/2}$, considered in the space $\overline{U}$, is bounded on the ellipsoid of the form $(u, u)_n \leqslant 1$ and therefore can be uniquely defined on the whole subspace $\overline{U}^+$ of the space $\overline{\overline{U}}$.

Let $u_1, u_2, ...$ be a complete system of eigenvectors of the operator

$$C = B^{-1/2} B_1 B^{-1/2}$$

in the Hilbert space $\overline{U}^+$, and let $\sigma_1^2, \sigma_2^2, ...$ be the system

corresponding eigenvalues. Without loss of generality, one may assume $a(u) \equiv 0$. Put

$$m_k = (u_k, B^{-1/2} a_1), \qquad v_k = B^{-1/2} u_k, \qquad k = 1, 2, ... \tag{5}$$

The density $p(x) = P_1(dx)/P(dx)$ can be expressed by the formula

$$\log p(x) = -\frac{1}{2} \sum_k \left\{ \log \sigma_k^2 + \frac{[v_k(x) - m_k]^2}{\sigma_k^2} - v_k(x)^2 \right\}, \tag{6}$$

where $v_k = v_k(x)$ are measurable functions of x corresponding to the elements $v_k \in \bar{U}$ $(k = 1, 2, ...)$.

This formula is easily obtained from general results, if one takes into account that the random variables $v_1, v_2, ...$ are independent both with respect to the measure $P(dx)$ and with respect to the measure $P_1(dx)$.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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CITED LITERATURE

1. I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Moscow, 1961.

2. Yu. V. Prohorov, Proc. IV Berkeley Symposium on Math. Statistics and Probability, 2, 1960.

3. Yu. A. Rozanov, *Theory of Probability and Its Applications*, 9, 3 (1964).

4. Yu. A. Rozanov, DAN, 165, No. 5 (1965).

5. C. R. Rao, V. S. Varadarajan, *Sankhyā*, A-25 (1963).

Note: Figure translations are in progress. See original paper for figures.