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Abstract

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MATHEMATICS

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ON FINITE GROUPS WITH A GIVEN SYLOW 2-SUBGROUP

(Presented by Academician A. I. Mal'cev on 18.IX.1965)

Gorenstein and Walter proved the following theorem ((2), Theorem 1).

Let G be a finite group containing a subgroup of order four that coincides with its centralizer in G . Let K be the largest normal divisor of odd order of the group G . Then one of the following assertions is true:

I. In the group G there is no subgroup of index 2, and a Sylow 2-subgroup S of the group G has the form

$$S = \langle \alpha, \beta \rangle, \quad \alpha^{2^{n+1}} = \beta^2 = 1, \quad \beta\alpha\beta = \alpha^{-1+2^n}, \quad n \geq 2. \quad (*)$$

II. The group G has an invariant series $G \supset G_0 \supset K \supset \{1\}$, where $[G : G_0] \leq 2$, $G_0/K \simeq PSL(2, q), PGL(2, q), SL(2, q)$ (q odd) or A_7 .

III. The factor group G/K is isomorphic to a Sylow 2-subgroup S of the group G .

In the present paper it is established that, in the case where assertion I of this theorem holds, $G/K \simeq PSL(3, 3)$ or M_{11} , the quadruple transitive Mathieu group of degree 11.

The structure is also established of finite groups whose Sylow 2-subgroups are of type (*) and in which the centralizer of the element $\tau = \alpha^{2^n}$ in the group G is soluble.

Theorem 1. *Let G be a finite group and let a Sylow 2-subgroup S of the group G have the form (). Let K be the largest normal divisor of odd order of the group G . If the centralizer $C_G(\tau)$ of the element $\tau = \alpha^{2^n}$ in the group G is soluble, then one of the following assertions is true:**

- 1) G has a normal series $G \supset G_0 \supset T \supset K$, where $[T : K] \leq 2$, $[G : G_0] = 2$, and G_0/T is isomorphic to a subgroup of $PGL(2, q)$ containing $PSL(2, q)$ (q odd). Here $PGL(2, q)$ is the group of all nonsingular projective semilinear transformations of the vector space of dimension 2 over a finite field of q elements.

2) $G/K \simeq PSL(3,3)$ or M_{11} .

3) G/K is isomorphic to a Sylow 2-subgroup S of the group G .

Proof. Let G/K not be isomorphic to S . If G contains a subgroup G_0 of index 2, then a Sylow 2-subgroup S_0 of the group G_0 is a dihedral or quaternion group. If S_0 is a dihedral group, then, by (3), G_0/K is isomorphic to a subgroup of $PGL(2, q)$ containing $PSL(2, q)$, or to the alternating group A_7 . Since the automorphism group of A_7 is isomorphic to S_7 , whose Sylow 2-subgroup is not of type (1), assertion 1) holds with $T = K$. If S_0 is a quaternion group, then (1) shows that $\bar{G}_0 = G_0/K$ has center \bar{T} of order 2 and in \bar{G}_0/\bar{T} the Sylow 2-subgroups are dihedral groups. In this case, by (3*), assertion 1) holds, where $[T : K] = 2$.

Suppose that in the group G there is no subgroup of index 2. By (4), $H = C_G(\tau)$ has the following structure: H contains a subgroup H_0 of index 2, in which there is no subgroup of index 2 and whose Sylow 2-subgroup is a quaternion group. We shall show that if A is the largest normal divisor of odd order of the group H , then $H/A \simeq GL(2,3)$. We may assume that $A = \{1\}$ and that in H there are no invariant subgroups of odd order. Consider-

we consider $\bar{H}_0 = H_0/\{\tau\}$. In \bar{H}_0 a Sylow 2-subgroup is a dihedral group and \bar{H}_0 has no nonidentity invariant subgroups of odd order, for otherwise H_0 , and with it also H , would have such subgroups. From (3) and the solvability of H it follows that $H_0 \cong PGL(2,3)$, whence it is easy to obtain that $H \cong GL(2,3)$. Thus, in G there is no subgroup of index 2 and $H/A \cong GL(2,3)$. In (4) such a situation was considered under the assumption that A is abelian. In this case $G/K \cong SL(3,3)$ or M_{11} . However, by slightly modifying the proof in (4), one can see that the requirement that A be commutative can be removed, which completes the proof of Theorem 1.

Theorem 2. *Let G be a finite group containing a subgroup of order four that coincides with its centralizer in G . Suppose G contains no subgroup of index 2. If a Sylow 2-subgroup S of G is of type (*), then $G/K \cong M_{11}$ or $PSL(3,3)$. Here K is the largest normal divisor of odd order of the group G .*

Proof. By (4), $H = C_G(\tau)$, where $\tau = a^{2^n}$, contains a subgroup H_0 of index 2, in which there is no subgroup of index 2 and a Sylow 2-subgroup of which is a quaternion group. Theorem 1 from (2) shows that if A is the largest normal divisor of odd order of the group H , then H_0/A is isomorphic to $SL(2, q)$, where q is odd. The element β induces in A a regular automorphism; otherwise $C(\{\tau, \beta\}) \neq \{\tau, \beta\}$, which contradicts the condition. Therefore A is an abelian subgroup, and for any element $\xi \in A$,

$$\beta\xi\beta = \xi^{-1}.$$

The element β induces in $\bar{H}_0 = H_0/A$ an automorphism of second order. Suppose that this automorphism leaves fixed an element ηA of the factor group H_0/A . Then $\beta\eta\beta = \eta\xi$, where $\xi \in A$. Since the order of A is odd, there exists

a natural number k such that $\xi^{2k} = \xi$. Then

$$\beta(\eta\xi^k)\beta = \eta\xi\xi^{-k} = \eta\xi^{2k}\xi^{-k} = \eta\xi^k,$$

and the element $\eta\xi^k \in C_H(\beta)$. Therefore $\eta\xi^k \in \{\tau\}$, and $\eta A = \{\tau A\}$. Thus β induces in $\bar{H}_0 = H_0/A$ an automorphism that fixes only the elements of the subgroup $\{\tau A\}$. We may assume $A = \{1\}$, otherwise all arguments can be carried out for H/A . We shall identify H_0 with $SL(2, q)$; $H = H_0\{\beta\}$, and β induces in $SL(2, q)$ an automorphism fixing only the elements of the cyclic subgroup

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

We first show that the order of a Sylow 2-subgroup S of the group H is 16. A Sylow 2-subgroup S_0 of the group H_0 is a maximal quaternion subgroup of some Sylow 2-subgroup S of the group H . It is easy to see that any element of $S \setminus S_0$ has order 2 or 2^{n+1} ; therefore any 2-element of $H \setminus H_0$ has order 2 or 2^{n+1} . In $SL(2, q)$ there exist cyclic subgroups X and Y of orders $q-1$ and $q+1$. Let $\{\rho\}$ be one of these subgroups. The subgroup $\beta\{\rho\}\beta$ belongs to $SL(2, q)$, and, since all cyclic subgroups of the same order are conjugate in $SL(2, q)$, there exists an element $\eta \in SL(2, q)$ such that

$$\beta\{\rho\}\beta = \eta\{\rho\}\eta^{-1}$$

and $\beta\eta \in N_H(\{\rho\})$. It is obvious that $\beta\eta \notin H_0$. Let $\beta\eta = \eta_1\eta_2 = \eta_2\eta_1$, where η_2 is a 2-element and the order of η_1 is odd. Since $[H : H_0] = 2$, $\eta_1 \in H_0$, and $\eta_2 \in H \setminus H_0$. Since η_2 is some power of the element $\beta\eta$, it follows that $\eta_2 \in N_H(\{\rho\})$. The order of η_2 is 2^{n+1} or 2.

Suppose the order of η_2 is 2^{n+1} . Then $\eta_2^2 \in N_{H_0}(\{\rho\})$, and from the properties of $SL(2, q)$ it follows that

$$\eta_2^4 = (\eta_2^2)^2 \in \{\rho\},$$

and in this case the order of $\{\rho\}$ is divisible by 2^{n-1} .

Suppose the order of η_2 is 2^{n+1} . Since $\eta_2^2 \in N_{H_0}(\{\rho\})$, and from the properties of $SL(2, q)$ are conjugate in H to β , we may assume that $\eta_2 = \beta$. Let $\rho = \rho_1\rho_2 = \rho_2\rho_1$, where ρ_2 is a 2-element and the order of ρ_1 is odd. Then $\beta \in N_H(\{\rho_1\})$, and, since the element β commutes with no nonidentity element of $\{\rho_1\}$,

$$\beta\rho_1\beta^{-1} = \rho_1^{-1}.$$

$\{\rho_2\}\{\beta\}$ belongs to some Sylow 2-subgroup of H and does not coincide with it. Therefore $\{\rho_2\}\{\beta\}$ is a dihedral group and

$$\beta\rho_2\beta = \rho_2^{-1}.$$

Hence

$$\beta\rho\beta^{-1} = \rho^{-1}.$$

But in $SL(2, q)$ there exists an element ζ such that

$$\zeta\rho\zeta^{-1} = \rho^{-1}$$

and $\zeta\beta \in C_G(\rho)$. Let $\zeta\beta = \xi_1\xi_2 = \xi_2\xi_1$, where ξ_2 is a 2-element and the order of ξ_1 is odd. $\xi_2 \in H \setminus H_0$ and $\xi_2\rho = \rho\xi_2$. If the order of ξ_2 is 2, then ξ_2 is conjugate to β , and we obtain that $\{\rho\} = \{\tau\}$, while since the order of $\rho \cdots$

is equal to $q-1$ or $q+1$, then $q=3$, and the order of a Sylow 2-subgroup S of the group H is 16. If, however, the order of ξ_2 is 2^{n+1} , then, as we have already shown, 2^{n-1} divides the order of $\langle\rho\rangle$. Thus, if $q \neq 3$, 2^{n-1} divides both $q-1$ and $q+1$, and hence their greatest common divisor is equal to 2, whence $n-1 \leq 1$, $n \leq 2$; but, by hypothesis, $n \geq 2$, and therefore in every case the order of S is 16.

The order of the group H is $2q(q-1)(q+1)$. If $q = r^2$, where r is an integer, then $q-1 = (r-1)(r+1)$ is divisible by 8, whence 32 divides the order of H . Therefore q is not a square, and the field $GF(q)$ has no automorphisms of order two. Every automorphism of $SL(2, q)$ is induced by a mapping $X \rightarrow AX^\sigma A^{-1}$, where $A \in GL(2, q)$, $X \in SL(2, q)$, and σ is an automorphism of the field $GF(q)$. Since $\beta^2 = 1$, for the corresponding automorphism we have $\sigma^2 = 1$; but, since q is not a square, $\sigma = 1$. The automorphism induced in $SL(2, q)$ by β is obtained by conjugating $SL(2, q)$ by a matrix $A \in GL(2, q)$ for which A^2 lies in the center

of $GL(2, q)$. If $q \equiv -1 \pmod{4}$, then as A one may choose the matrix $\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$,

where $i^2 = -1$, $i \in GF(q)$, but it commutes with matrices of the form $\begin{pmatrix} x & y \\ iy & x \end{pmatrix}$, while the equation $x^2 - iy^2 = 1$ has in $GF(q)$ solutions other than $x = \pm 1$, $y = 0$.

Therefore $q \equiv 1 \pmod{4}$. As A one may choose the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and it

commutes with matrices in $SL(2, q)$ of the form $\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$, where $v \in GF(q)$,

$v \neq 0$. The number of such matrices is $q-1$. Therefore $q-1 = 2$ and $q = 3$. We have shown that the centralizer of the element τ is solvable. Now Theorem 1 gives $G/K \simeq PSL(3, 3)$ or M_{11} .

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Note: Figure translations are in progress. See original paper for figures.

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