FURTHER ON THE BREAKING OF WAVES IN SHALLOW WATER

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Abstract

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FURTHER ON THE BREAKING OF WAVES IN SHALLOW WATER

Our work ([1]), devoted to the question of the breaking of waves in shallow water, was published 12 years ago. In it we revealed errors introduced at one time by preceding authors into the analysis of phenomena occurring during the propagation of tidal waves (([2]; ([3]), p. 165) and surface waves (([4]; ([2]), p. 248) in shallow regions of the ocean or sea. It was shown that such errors do not arise if one takes into account the natural consequence of the equation of continuity, which is written quite correctly in works ([2-4]) as applied to sea depths H comparable with the deviations η of the sea surface from the level of rest:

$$\frac{\partial}{\partial x}[(H+\eta)u] = -\frac{\partial \eta}{\partial t}.$$

Here u is the horizontal component of the velocity of particles in a wave propagating along the x-axis; t is the current time. It is quite obvious that the velocity of level oscillations here depends on the phase (in parentheses stands the algebraic sum $H+\eta$); consequently, Euler' s equation together with such an equation of continuity must give a solution describing a wave whose phase velocity depends on the phase.

All the difficulties that hinder the integration of manifestly nonlinear partial differential equations stand in the way of such a solution. Therefore, in 1954 we applied an indirect method proceeding from a natural hypothesis: instantaneous values of the phase velocities are, with sufficient approximation, determined by the classical formulas, in which, instead of the constant sea depth H, the corresponding instantaneous values $H+\eta$ should be substituted. In the most general case they enter the argument $2\pi(H+\eta)/\lambda$ of the hyperbolic tangent.

The analysis described in ([1]), and also in ([5]), made it possible to construct profiles of waves gradually distorted under the influence of shallow water, for any initial profile and any distance traversed in shallow water. The calculated profiles agreed well with actually observed ones, both in the case of a dead swell that initially had an almost trochoidal form and in the case of a wind wave that initially possessed the well-known sharp crest at the top (of the Michell type). The gradual changes of a simple sinusoidal profile were studied in detail (Fig. 83

in ([5])), up to the stage preceding the breaking of the crest. It was shown that all the profiles obtained, when expanded in a Fourier sine series, are characterized by a continuous decrease in the amplitude of the fundamental sinusoid and by a gradual increase of the overtones (Fig. 84 of work ([5])), traced up to the 16th.

In the present article an analytical confirmation is given of the hypothesis that has already passed experimental verification.

We shall confine ourselves only to the case of propagation of a very gently sloping wave (a tidal wave or a low, very long dead swell), whose initial profile may be regarded as sinusoidal. One may obviously neglect the term $u\,\partial u/\partial x$ in Euler' s equation and the term $u\,\partial \eta/\partial x$ in the expanded equation of continuity. By the usual device of repeated differentiation, u is eliminated and one equation arises:

$$\partial^2\eta/\partial t^2 - (H+\eta)^{-1}(\partial\eta/\partial t)^2 = g(H+\eta)\,\partial^2\eta/\partial x^2, \tag{1}$$

valid under the condition $H \ll \lambda$, where λ is the wavelength. As we see, here $g(H+\eta)$ takes the place of the square of the phase velocity in the usual wave equation, qualitatively justifying our hypothesis.

Let us pass to dimensionless variables

$$y = \eta/H; \qquad s = x/H; \qquad \tau = t\sqrt{g/H}. \tag{2}$$

Then, instead of (1), we obtain

$$\partial^2 y/\partial\tau^2 - (1+y)^{-1}(\partial y/\partial\tau)^2 = (1+y)\partial^2 y/\partial s^2. \tag{3}$$

In general form this equation is not integrable. For numerical integration by any means it is necessary to specify concrete parameters.

On the basis of ([1,5]), the critical profile of a distorted wave with a vertical front should arise under the condition $x/\lambda = 2H/3h$, where $h = 2a$ denotes the height (twice the amplitude) of the wave. Suppose such a profile arises after the wave has traveled from the boundary of the shallow-water region a distance $x = 4\lambda$. Then it must be that $H/h = 6$. Let us set $H = 10$ m. Then it follows that $h = 1.66$ m; $a = 0.83$ m. On the basis of (2), the dimensionless amplitude of the waves will be $a/H = y_0 = 0.083$.

In accordance with the order of the periods of real very long waves of swell (about 1 min), let us take for the dimensionless wave period

$$\mathcal{T} = T\sqrt{\frac{g}{H}} = \frac{\lambda}{\sqrt{gH}}\sqrt{\frac{g}{H}} = \frac{\lambda}{H}$$

the value $\mathcal{T} = 20\pi$. This corresponds to a swell length of more than 600 m. Thus, $H \ll \lambda$.

With these specified parameters, regular sinusoidal waves arriving from the deep-water region of the sea should, at the very beginning of the shallow water, produce level oscillations described by the equation

$$y = 0.083 \sin 0.1\tau. \tag{4}$$

On the basis of ([1,5]), we assume that the level oscillations on the wave occurring at a distance s from the boundary of the shallow-water region follow the law:

$$y = \sum_{n=1}^{\infty} B_n(s) \sin 0.1\, n(\tau - s). \tag{5}$$

Here the coefficients $B_n(s)$ of the series depend only on the dimensionless distance s traveled in shallow water. This dependence is easily derived from the data of ([1,5]) as applied to the accepted parameters of the problem.

If expression (5) were ideally exact, then substitution of the expression y and of the corresponding derivatives with respect to τ and s into (3) would lead to an identity between the left- and right-hand sides of (3) in all phases of the level oscillation at the specified distance s from the boundary of the shallow-water region.

In reality, complete identity cannot be expected, but it is possible to analyze the results by applying the method that enabled N. E. Kochin for the first time to obtain, from nonintegrable equations of hydrodynamics, a simplified system of equations describing the general circulation of the atmosphere with sufficient accuracy ([6])—the method of estimating the errors introduced by the simplifications.

On the basis of ([5]), the following will enter the left-hand side of (3):

$$\frac{\partial^2 y}{\partial \tau^2} = -\sum_{n=1}^{\infty} 0.01\, n^2 B_n \sin 0.1\, n(\tau - s), \tag{6}$$

$$\left(\frac{\partial y}{\partial \tau}\right)^2 = \left[\sum_{n=1}^{\infty} 0.1\, n B_n \cos 0.1\, n(\tau - s)\right]^2, \tag{7}$$

where (7) must also be divided by the value $(1 + y)$ in each phase.

On the right-hand side there will be

$$\frac{\partial^2 y}{\partial s^2} = \sum_{n=1}^{\infty} \frac{\partial^2 B_n}{\partial s^2} \sin 0.1n(\tau - s) - 2\sum_{n=1}^{\infty} 0.1n \frac{\partial B_n}{\partial s} \cos 0.1n(\tau - s) - \sum_{n=1}^{\infty} 0.01n^2 B_n \sin 0.1(\tau - s). \tag{8}$$

Fig. 1

Figure 1: Fig. 1

This expression must be multiplied by $(1+y)$, computed for each phase. For the given parameters, the vertical front of the distorted wave before the breaking of its crest should arise at the dimensionless distance $s_{\text{crit}} = 4 \cdot 20\pi = 251$ from the boundary of the shallow water. But here all the series (6)–(8) become divergent, since all the derivatives tend to infinity. Therefore we shall carry out computations for the profile of waves that have traversed half the critical path, i.e. $s = 126$. Here it will be possible to restrict oneself to summing 7 terms of the series in (6) and (8), and 8 terms of the series in (7). Even so, small serrations remain on the curves computed from (6)–(8); they have no real meaning and are quite insignificant. Therefore, as a check, along with the computation by (7), a computation was made by directly finding the derivative with respect to the given distorted profile on the path halfway to breaking.

Fig. 1

In Fig. 1, curve 2 represents this profile, constructed from (1, 5) for $s = 126$. Curve 1 is the initial profile of the waves that have just entered the shallow water, and 3 is the profile of the waves before the breaking of their crests ($s_{\text{crit}} = 251$). Both methods gave close results.

In Fig. 1 two curves are superposed: 4 represents the variations of the left-hand side, and 5 the right-hand side of (3). The discrepancies between these curves are relatively small. Their maximum ordinates differ by approximately 5%. The phase shift near passage through zero is also insignificant.

After the derivation set out above had already been submitted to the editorial office of DAN, at Moscow University the integration of our equation (3) on an electronic computer, organized by A. N. Tikhonov, was completed. B. I. Volkov succeeded in compiling a program and obtaining on the machine the law of oscillations of the level on the wave for two values of the dimensionless distance from the edge of the shallow water. The points obtained by him are plotted in Fig. 2 as circles, through which curve 1 is drawn for the value $s = \frac{3}{4} s_{\text{crit}} =$

$= 188$ and curve 2 for the value $s_{\text{crit}} = 251$. The diagram makes it possible to estimate the errors introduced by our hypothesis even better than Fig. 1.

On the basis of the hypothesis, the shift of the points of the distorted profile relative to the original sinusoid 0 must be proportional to the distance of the corresponding point along the vertical from the zero level. In full agreement with this, straight lines 3-10 represent the geometric loci of the ends of the segments expressing the displacements; the beginnings of the segments are taken on ordinates passing through the crest and through the trough of sinusoid 0. Straight line 3,

Fig. 2

Fig. 2

Figure 2: Fig. 2

is constructed on the basis of (1, 5). Straight line 5 corresponds to the initial segment of curve 1; 4—to the segment from its crest to the abscissa axis, and 9—to the entire lower part of curve 1. Similarly, straight line 6 is constructed on the basis of (1, 5) for the critical value of s; 8 corresponds to the initial segment of curve 2; 7—to its segment from the crest to the abscissa axis; 10—from the abscissa axis to the trough of the wave. Further, a scatter of points is observed, apparently caused by the features of equation 3 at the critical value of s. The general conclusion from the constructions is as follows: on the rising initial segments the machine gave points that correspond to distances s 10% smaller than those obtained on the basis of (1, 5); on the descending segments down to the abscissa axis the difference is only 2%; the sections of the curves below the abscissa axis (where there is no scatter of points) agree perfectly accurately with the hypothesis on the basis of which straight lines 9 and 10 were drawn. The continuity condition does not allow differences in the displacement of points on the rising and, correspondingly, descending segments of the distorted wave profile at unchanged wave height. Consequently, the probable error introduced by the hypothesis cannot exceed the half-sum of 10 and 2%. The absence of errors in the segments below the abscissa axis shows that the probable error is even less than 6%.

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Note: Figure translations are in progress. See original paper for figures.