THE BEHAVIOR OF METRIZABILITY UNDER QUOTIENT MONOTONE MAPPINGS

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THE BEHAVIOR OF METRIZABILITY UNDER QUOTIENT MONOTONE MAPPINGS

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One of the problems of set-theoretic topology is to determine conditions for the preservation of metrizability under continuous mappings of various kinds. In this form the problem is still very general; it is not surprising that the first result pertaining to it had a rather special character: the image of a metrizable space under a perfect mapping is metrizable. This assertion was proved by Stone and Morita jointly with Hanai, and earlier, for spaces with a countable base, by Vainstein. Then there arose the much more general problem of determining conditions for the preservation of metrizability under arbitrary quotient mappings. A complete solution of this problem was proposed by A. Arhangel' skii ([2]); he also obtained in this area a number of special results ([2,3]).

American mathematicians Paul McDougall and A. Martin worked in the same area. The most natural restrictions on the elements of decompositions are usually the requirements of connectedness and bicompactness (mappings corresponding to such decompositions are called, respectively, monotone and bicompact). It turns out, however, that in the general case monotonicity gives nothing.

The following fact holds: if the space Y is a pseudo-open bicompact image of a metric space X, then the space Y is also a pseudo-open, bicompact, and monotone image of some other metric space Z.

We shall prove this. Let $f : X \to Y$ be some pseudo-open bicompact mapping of a metric space. Take the decomposition $\{A_y \mid y \in Y\}$ of the space X, where $A_y = f_y^{-1}$, and consider the system of sets $\{B_y \mid y \in Y\}$, where $B_y = A_y \times I$. Let $\rho(x, x')$ be the metric in X; $d(r, r')$ the metric in $I = [0, 1]$.

Put $X_1 = \bigcup_{y \in Y} B_y$. On X_1 we introduce a topology: a) an arbitrary neighborhood of the point $(x, 0)$ is defined as

$$O_{(x,0)} = O_x \times [0, \varepsilon),$$

where O_x is some neighborhood of the point x in the space X and ε is any positive number; b) as neighborhoods of points $(x, r) \in B_{y_0}$ (where $r > 0$) we shall call sets of the form

$$O_{(x,r)} = O_{[(x,r),\varepsilon]} \cap B_{y_0},$$

where $O_{[(x,r),\varepsilon]}$ is the ε-neighborhood of the point $(x,r) \in X_1$ in the sense of the metric

$$\rho^*((x,r),(x',r')) = \sqrt{\rho(x,x')^2 + d(r,r')^2}$$

and $0 < \varepsilon < r$. The space X_1 with this topology is regular, and every set $B_y \subset X_1$ is a compact. It is easy to see that, for any n, the set

$$B_y^n = A_y \times (1/n, 1] \subset B_y$$

is open in X_1. We show that in the space X_1 there exists a σ-discrete base. Let

$$\pi = \{\gamma_i = \{U_\beta^i\} \mid i = 1, 2, ...\}$$

be some σ-discrete base of the space X. The systems

$$\omega_{in} = \{V_\beta^{in}\} \quad (i, n = 1, 2, ...),$$

where

$$V_\beta^{in} = U_\beta^i \times [0, 1/n],$$

are discrete in X_1, and moreover

$$\Omega_1 = \bigcup_{i,n} \omega_{in}$$

forms a base at all points of the form $(x, 0)$. Further, let

$$d_{yn} = \{\xi_j^{ny}\}$$

be a σ-discrete base in B_y^n. Since the system of open sets $\{B_y^n \mid y \in Y\}$ is discrete in X_1, and the system ξ_j^{ny} is discrete in B_y^n, the system

$$\xi_j^n = \bigcup_{y \in Y} \xi_j^{ny}$$

is discrete in X_1.

The system $\Omega_2 = \bigcup_{j,n} \xi_j^n = \bigcup_{n,y\in Y} d_y^n$ forms a base at all points of $X_1 \setminus (X \times \{0\})$. Hence, $\Omega = \Omega_1 \cup \Omega_2$ is a σ-discrete base of the space X_1. We conclude, by Bing' s theorem (7), that the space X_1 is metrizable. Put $Z = (X \times [0, 1)) \cup Y$; define the mapping $g : X_1 \to Z$ as follows:

$$g[(x,r)] = \begin{cases} (x,r), & \text{if } r < 1, \\ y, & \text{if } r = 1. \end{cases}$$

On the set Z introduce the quotient topology corresponding to the mapping g. It is not hard to verify that the mapping g is perfect; therefore the space Z is metrizable. We construct a mapping $\tilde{f} : Z \to Y$ by putting $\tilde{f}z = y$, if $g^{-1}z \subset B_y$. The mapping is monotone and bicompact. Since $f = \tilde{f}|g(X \times \{0\}) \equiv X$, the mapping $\tilde{f}$ is pseudo-open. In general, under quotient monotone and bicompact mappings of metric spaces even the first axiom of countability need not be preserved. This is shown by the following well-known example. Take Hilbert space $H^{\aleph_0}$. Define a certain decomposition of it. The elements of this decomposition are defined as follows: first, the intervals $[(0, \dots, 1/(n+1), 0, \dots), (0, \dots, 0, 1, 0, \dots)] \in L_n$ $(n = 1, 2, \dots)$, where L_n is the n-th coordinate axis in $H^{\aleph_0}$, and, second, single points. Endow the set of elements of the decomposition with the quotient topology. This quotient topology does not satisfy the first axiom of countability, although the corresponding quotient mapping is monotone and bicompact. A. Martin proved the following assertion (see (11)):

If $f : X \to Y$ is a quotient bicompact and monotone mapping of a locally bicompact metric space X onto a Hausdorff space Y, then Y is metrizable and locally bicompact.

Theorem 1, which is the main result of the present paper, considerably generalizes Theorem 5 of A. Arhangel' skii (see (2)) and, in addition, overlaps one theorem of A. Martin.

Theorem 1. Let $f : X \to Y$ be a quotient monotone mapping of a completely paracompact* metric space X onto a regular space Y of point-countable type**. Then Y is metrizable.

In Theorem 1 the monotonicity of the given mapping cannot be omitted, since from results of V. Ponomarev (8) it follows that every space Z with a point-countable base is the open s-image of some strongly paracompact metric space. This shows that Theorem 1 is, in a certain sense, a definitive result.

Lemma 1. Let $f : X \to Y$ be a quotient monotone mapping of a topological space X onto a finally compact space Y. If in X there exists a base $\omega = \{\gamma_n\}$ consisting of a countable number of star-countable covers, then there also exists a countable base in X.

Proof. It is enough to prove that each γ_n consists of no more than a countable number of elements. Since the cover γ_n is star-countable, on the basis of the well-known theorem of P. S. Aleksandrov (1), the space X decomposes into open-and-closed sets $\{M_\alpha, \alpha \in \Theta\}$, where each M_α is the sum of a countable number of elements of γ_n. The system $\{M_\alpha, \alpha \in \Theta\}$ is discrete in X, and, by virtue of the quotientness and monotonicity of the mapping f, the system $\{fM_\alpha \mid \alpha \in \Theta\}$ is also discrete in Y and covers all of Y (it is essential that $M_\alpha = f^{-1}fM_\alpha,\ \alpha \in \Theta$). Since Y is finally compact,

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* A metric space X is completely paracompact if it has a base consisting of a

countable number of star-finite covers.

** A space X is called a space of point-countable type if every point $x \in X$ is contained in some bicompact $\Phi \subset X$ whose character in X is countable.

space, and $\{M_\alpha \mid \alpha \in \theta\}$ is its discrete open cover, then θ is at most countable. Hence the cover γ_n is also at most countable.

Lemma 2. If a bicompactum Φ has countable character in a T_2-space X, and the point $x_0 \in \Phi \subset X$ has countable character in Φ, then the point x_0 has countable character also in the space X (see ([10])).

Proof of Theorem 1. We shall prove first that Y satisfies the first axiom of countability. Take an arbitrary point $y_0 \in Y$. By hypothesis, there exists a bicompactum $\Phi_0 \subset Y$ of countable character such that $y_0 \in \Phi_0$. It follows from Lemma 1 that $X_1 = f^{-1}\Phi_0$ has a countable base, and then, by a theorem of A. Arhangel' skii (see ([6])), the bicompactum Φ_0 is metrizable (it has a countable net by virtue of the continuity of the mapping $f_1 = f|X_1$). Then, by Lemma 2, the character of the point y_0 in the space Y is countable. We now prove that there exists a σ-discrete base in Y. Let $\{\gamma_n\}$ be some system of star-finite covers of the space X forming a base for it. Suppose that, for every n, the cover γ_{n+1} is inscribed in γ_n. The space X, on the basis of a theorem of P. S. Aleksandrov, decomposes, with respect to the cover γ_1, into open-and-closed pieces M_{λ^1}, $\lambda^1 \in \theta_1$, where each M_{λ^1} is the sum of a countable number of elements of γ_1. Since the cover γ_n $(n > 1)$ is inscribed in γ_1, and the system $\sigma' = \{M_{\lambda^1}, \lambda^1 \in \theta_1\}$ is discrete in X, no element of the cover γ_n intersects simultaneously two elements of the system σ^1. It follows from this that each set $M_{\lambda^1} \in \sigma^1$ decomposes into open-and-closed pieces $M_{\lambda^1\lambda^2}$, $\lambda^2 \in \theta_2$, where each $M_{\lambda^1\lambda^2}$ is the sum of at most a countable number of elements of the cover γ_2. Put

$$\sigma^2 = \{M_{\lambda^1\lambda^2}, \lambda^1 \in \theta, \lambda^2 \in \theta_2\}.$$

Continuing to argue in this way by induction, for each number n we construct a discrete system

$$\sigma^n = \{M_{\lambda^1\dots\lambda^n}, \lambda^1 \in \theta_1, \dots, \lambda^n \in \theta_n\}$$

with the following properties: a) each $M_{\lambda^1\dots\lambda^n}$, open-and-closed in X, is the sum of at most a countable number of elements of γ_n; b)

$$M_{\lambda^1} \supset M_{\lambda_0^1\lambda^2} \supset \dots \supset M_{\lambda_0^1\lambda_0^2\dots\lambda_0^n}.$$

Let

$$\omega_k = \bigcup_{i=1}^{k} \gamma_i = \{U_\alpha^k\}.$$

Define a system $\{\omega'_k\}$ of covers in the following way: put $\omega'_k = \{V^k_\beta\}$, where each V^k_β is the intersection of some element of ω_k with some element of σ^k. Obviously,

$$\omega'_k = \bigcup_{\lambda^i \in \theta_i} d^k_{\lambda^1 \dots \lambda^k}, \dots, \qquad d^k_{\lambda^1 \dots \lambda^k} = \{V^k_\alpha \mid V^k_\alpha = U^k_\alpha \cap M_{\lambda^1 \dots \lambda^k}\}.$$

By virtue of conditions a) and b), the set of nonempty elements of the system $d^k_{\lambda^1 \dots \lambda^k}$ is at most countable. Define the discrete systems

$$\Omega^i_k = \{\Gamma^{ki}_{\lambda^1 \dots \lambda^k}, \lambda^1 \in \theta_1, \dots, \lambda^k \in \theta_k\},$$

where each $\Gamma^{ki}_{\lambda^1 \dots \lambda^k}$ is the union of a finite number of nonempty elements of $d^k_{\lambda^1 \dots \lambda^k}$. By virtue of the quotientness and monotonicity of the mapping f, the cover $f\sigma^k$ is discrete in Y, and therefore the system $f\Omega^i_k$ is discrete in Y.

Put

$$f\widetilde{\Omega}^i_k = \{\operatorname{Int} f\Gamma^{ki}_{\lambda^1 \dots \lambda^k}\}.$$

We shall show that the sets of the system

$$\mathfrak{A} = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} f\widetilde{\Omega}^i_k$$

form a σ-discrete base in the space Y. Choose an arbitrary point $y_0 \in Y$ and its neighborhood O_{y_0}. By the monotonicity of the mapping f, the set $f^{-1}y_0$ intersects only one set of the system

$$\sigma^k = \{M_{\lambda^1, \dots, \lambda^k}\} \quad (k = 1, 2, \dots),$$

i.e. no more than a countable aggregate of elements of

$$\bigcup_{k=1}^{\infty} \omega_k.$$

Then there exists such a countable system

$$\eta_{y_0} = \{U^{k_1}_{\alpha_1}, \dots, U^{k_s}_{\alpha_s}, \dots\},$$

that $U^{k_s}_{\alpha_s} \in \omega_{k_s}$, $U^{k_s}_{\alpha_s} \subset f^{-1}O_{y_0}$ and

$$\bigcup_{s=1}^{\infty} U^{k_s}_{\alpha_s} \supset f^{-1}y_0.$$

Let

$$G_i = \bigcup_{s=1}^{i} U^{k_s}_{\alpha_s}.$$

Obviously,

$$\bigcup_{i=1}^{\infty} G_i \supset f^{-1}y_0, \qquad G_i \subset f^{-1}O_{y_0}$$

and

$G_{i+1} \supset G_i$ $(i = 1, 2, \ldots)$. By Stone' s lemma ([9]), there exists a number n_0 such that $\operatorname{Int} fG_{n_0} \ni y_0$. Put $k_0 = \max(k_1, \ldots, k_{n_0})$; then $f^{-1}y_0 \in M_{\lambda_0^1 \ldots \lambda_1^{k_0}}$, and $U_{\alpha_s^1} \cap M_{\lambda_0^1 \ldots \lambda_0^{k_0}}$ (where $1 \leq s \leq n_0$) is some $V_{\alpha_s}^{k_0} \in d^{k_0}_{\lambda_0^1 \ldots \lambda_0^{k_0} j_s}$, and

$$\bigcup_{s=1}^{n_0} V_{\alpha_s}^{k_0} = \Gamma^{k_0 j_0}_{\lambda_0^1 \ldots \lambda_0^{k_0}}.$$

Obviously,

$$\Gamma^{k_0 j_0}_{\lambda_0^1 \ldots \lambda_0^{k_0}} = G_{n_0} \cap M_{\lambda_0^1 \ldots \lambda_0^{k_0}}.$$

From the fact that $M_{\lambda_0^1 \ldots \lambda_0^{k_0}} = f^{-1} f M_{\lambda_0^1 \ldots \lambda_0^{k_0}}$ and that $fM_{\lambda_0^1 \ldots \lambda_0^{k_0}}$ is open in Y, it follows that

$$\operatorname{Int} f\Gamma^{k_0 j_0}_{\lambda_0^1 \ldots \lambda_0^{k_0}} = fM_{\lambda_0^1 \ldots \lambda_0^{k_0}} \cap (\operatorname{Int} fG_{n_0}).$$

Thus $y_0 \in \operatorname{Int} f\Gamma^{k_0 j_0}_{\lambda_0^1 \ldots \lambda_0^{k_0}}$. Since $\operatorname{Int} f\Gamma^{k_0 j_0}_{\lambda_0^1 \ldots \lambda_0^{k_0}} \in f\widetilde{\Omega}^{j_0}_{k_0}$, this proves that the σ-discrete system $\mathfrak{A}$ is a base in the space Y; consequently, by Bing' s criterion ([7]), the space Y is metrizable.

Since under almost open and inductively open mappings (see ([3])) the first axiom of countability is preserved, Theorem 1 implies the following assertion:

Corollary. *Let $f : X \to Y$ be an inductively open (or almost open) monotone mapping of a locally compact metric space X onto a regular space Y. Then Y is metrizable.*

The last result is also new for open mappings.

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Note: Figure translations are in progress. See original paper for figures.