TOPOLOGY OF FUNCTION SPACES

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.836 + 519.3

MATHEMATICS

S. I. AL' BER

TOPOLOGY OF FUNCTION SPACES

(Presented by Academician P. S. Aleksandrov on 28 IX 1965)

The investigation of the topological properties of function spaces, chiefly in connection with the solution of problems of the calculus of variations as a whole, has been carried out in a number of works. The homologies of the spaces of oriented loops $\Omega(S^n)$ and oriented closed paths $\Pi(S^n)$ on the sphere S^n have been completely computed, and the rational homologies have also been computed for the spaces of unoriented closed paths $\hat{\Pi}(S^n)$, of oriented and unoriented closed curves $K(S^n)$ and $\hat{K}(S^n)$. In ([2]) the mod 2 homologies of the space $\hat{\Omega}(S^n)$ of unoriented loops were investigated. Here we shall completely compute the homologies of the space of unoriented loops $\hat{\Omega}(M)$ and the space of unoriented closed paths $\hat{\Pi}(M)$ on a Riemannian manifold diffeomorphic to the sphere S^n, and also give some results on the typical numbers of closed geodesics. We use the definitions and notation of the papers ([2],[3]).

1. Grading of homology groups

In computing homologies by the variational method, the homology groups naturally decompose into series of groups corresponding to different critical values.

Definition 1. If in the function space L a functional $J(u)$, where $u \in L$, is defined, and a contracting deformation $D_t : L \to L$ is constructed (see ([3])), then the **critical value of the class of singular homologies** $\{Z\}$ is called the number

$$q = \inf_{Z\in\{Z\}} \left(\lim_{t\to\infty} \max_{u\in Z} J[D_t(u)] \right). \tag{1}$$

Choose in the group $H_s(L, A)$ a complete independent system of generators, and denote by $H_{s,q}(LA)$ the subgroup generated by generators with critical value q. Then

$$H_s(L, A) = \sum_q H_{s,q}(L, A). \tag{2}$$

2. Integral homologies of the space of unoriented loops

The cycles $[j, 2k-1]$ constructed in (^{2}) will henceforth be denoted by $[j, k]$.

Theorem 1. The integral homology groups $H_{s,q}(\hat{\Omega}(S^n), Z)$ ($q = 2\pi k$; $k = 0, 1, 2, \ldots$) are isomorphic to:

1) If $n = 2m-1$ is odd:

$$H_{s,q}(\hat{\Omega}(S^n), Z) = \begin{cases} Z & \text{for } s = 2k(n-1), \\ Z_2 & \text{for } s = (2k-1)(n-1) + 2j, \text{ where } j = 0, 1, \ldots, m-2, \\ 0 & \text{in other cases.} \end{cases} \tag{3}$$

The cycles $[2j, k]$ ($j = 0, 1, \ldots, m-1$) form a basis system.

2) If $n = 2m$ is even and k is even:

$$H_{s,q}(\hat{\Omega}(S^n), Z) = \begin{cases} Z & \text{for } s = 2k(n-1) \text{ and } s = (2k-1)(n-1), \\ Z_2 & \text{for } s = (2k-1)(n-1) + (2j+1), \text{ where } j = 0, 1, \ldots, m-2, \\ 0 & \text{in other cases.} \end{cases} \tag{4}$$

The cycles $[0, k]$ and $[n-1, k]$ are free generators, while the cycles $[2j+1, k]$ form a basic system of torsion groups.

3) If $n = 2m$ is even, but k is odd:

$$H_{s,q}(\hat{\Omega}(S^n), Z) = \begin{cases} Z_2 & \text{for } s = (2k-1)(n-1) + 2j, \text{ where } j = 0, 1, \ldots, m-1, \\ 0 & \text{in other cases.} \end{cases} \tag{5}$$

The cycles $[2j, k]$ form a basic system of torsion groups.

For the proof, consider the manifold $[n-1, k] = \hat{Q}^{2k(n-1)}$ of all $(2k-1)$-link unoriented loops, which, as can be shown, forms a transverse index stratification in a neighborhood of the extremal manifold P_k^{n-1}. Therefore the computation of the groups $H_{s,q}(\hat{\Omega}(S^n), Z)$ reduces to the study of the groups $H_{s,q}(\hat{Q}^{2k(n-1)}, Z)$. Let $Q^{2k(n-1)}$ be the covering of $\hat{Q}^{2k(n-1)}$ consisting of all oriented $(2k-1)$-link loops.

The involution $\Omega(S^n) \xrightarrow{\theta} \Omega(S^n)$ induces an involution in the cellular complex $Q^{2k(n-1)} = S_0^{n-1} \times \cdots \times S_{2k-1}^{n-1}$. If $[j, k]_\Omega$ is the covering of the cycle $[j, k]$ in the space $\Omega(S^n)$, then the action of the involution is given by the formula

$$\theta([j,k]_\Omega) = (-1)^{(n-1)(k-1)+j}[j,k]_\Omega, \tag{6}$$

from which the proof of the theorem follows.

3. Homologies of the space of unoriented closed paths. A closed path $x(t)$ $(t = \varphi/2\pi,\ 0 \le \varphi \le 2\pi)$ of a Riemannian manifold M^n is a continuous mapping of the circle with a distinguished initial point into the manifold

$$x : S^1 \to M^n. \tag{7}$$

The space of oriented closed paths of the manifold M^n is denoted by $\Pi(M^n)$. In the space $\Pi(M^n)$ there acts an involution

$$\Pi(M^n) \xrightarrow{\theta} \Pi(M^n) \tag{8}$$

according to the formula

$$x_2(t) = (\theta x_1)(t) = x_1(1-t), \qquad 0 \le t \le 1. \tag{9}$$

Identifying closed paths satisfying condition (9), we obtain the space $\hat{\Pi}(M^n)$ of unoriented closed paths. Note that $\hat{\Pi}(M^n)$ is a fibration with base M^n and fiber $\hat{\Omega}(M^n)$. On the sphere S^n the closed geodesic paths coincide with multiple great circles (with distinguished points). Therefore the set of extremals consists of a series of manifolds $\hat{T}_0^n$, $\hat{T}_1^{2n-1}, \dots, \hat{T}_k^{2n-1}, \dots$, where the manifold $\hat{T}_0^n$ is diffeomorphic to the sphere S^n, and the manifolds $\hat{T}_k^{2n-1}$ are diffeomorphic to the manifold T^{2n-1} of unoriented tangent elements of the sphere S^n.

Let us construct cycles of the manifold $\hat{T}^{2n-1}$. Let m_0 be the base point of the sphere S^n. The unoriented tangents at the point m_0 form a submanifold P^{n-1}, diffeomorphic to projective space. The cycles $[i]$ $(i = 0, 1, \dots, n-1)$ of this space will be cycles of the manifold $\hat{T}^{2n-1}$. Choose an arbitrary cycle $[i]$ and consider the set of circles of the sphere S^n passing through the point m_0, whose tangents at the point m_0 belong to the cycle $[i]$. At each point of the circle draw an unoriented tangent. The resulting set of tangents forms the cycle $[i+n]$ of the manifold $\hat{T}^{2n-1}$.

Theorem 2. The cycles $[j]$ $(j = 0, 1, \dots, 2n-1)$ form a complete basic system of cycles mod 2 of the manifold $\hat{T}^{2n-1}$. The integral homology groups of the manifold $\hat{T}^{2n-1}$ are isomorphic:

1) If n is odd:

$$H_s(\hat{T}^{2n-1}) = \begin{cases} H_s(P^{n-1}) & \text{for } 0 \leqslant s \leqslant n-1, \\ H_{s-n}(P^{n-1}) & \text{for } n \leqslant s \leqslant 2n-1. \end{cases} \tag{10}$$

2) If n is even:

$$H_s(\hat{T}^{2n-1}) = \begin{cases} H_s(P^{n-1}) & \text{for } 0 \leqslant s \leqslant n-2, \\ Z_4 & \text{for } s = n-1, \\ 0 & \text{for } s = n, \\ H_{s-n}(P^{n-1}) & \text{for } n+1 \leqslant s \leqslant 2n-1. \end{cases} \tag{11}$$

Let $[j]_k$ $(j = 0, 1, \dots, 2n-1)$ be the cycle of the extremal manifold $\hat{T}_k^{2n-1}$, obtained as the image of the cycle $[j]$ under the natural diffeomorphism $\hat{T}^{2n-1} \to \hat{T}_k^{2n-1}$. The set of $(2k-1)$-link unoriented loops with supporting circles belonging to $[j]_k$, and with base points—marked points on these circles—forms a cycle $[j, k]$ modulo 2 of the space $\hat{\Pi}(S^n)$.

Theorem 3. The homology groups $H_{s,q}(\hat{\Pi}(S^n), Z_2)$ $(q = 2\pi k;\ k = 0, 1, \dots)$ are isomorphic to:

$$H_{s,q}(\hat{\Pi}(S^n), Z_2) = \begin{cases} Z_2 & \text{for } k = 0 \text{ and } s = n, \\ Z_2 & \text{for } (2k-1)(n-1) \leqslant s \leqslant (2k+1)(n-1)+1, \\ & \text{if } k > 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{12}$$

The cycles $[j, k]$ $(j = 0, 1, \dots, 2n-1;\ k = 1, 2, \dots)$ and $[n, 0]$ form a complete basic system of cycles modulo 2 of the space $\hat{\Pi}(S^n)$.

Theorem 4. The integral homology groups of the space $\hat{\Pi}(S^n)$ of unoriented closed paths are isomorphic to:

1) If either n or k is odd:

$$H_{s,q}(\hat{\Pi}(S^n), Z) = \begin{cases} Z & \text{for } k = 0 \text{ and } s = n, \\ H_{s,q}(\Omega(S^n), Z) & \text{for } (2k-1)(n-1) \leqslant s \leqslant 2k(n-1), \\ & k > 0, \\ H_{s-n,q}(\Omega(S^n), Z) & \text{for } 2k(n-1) \leqslant s \leqslant \\ & \leqslant (2k+1)(n-1)+1,\ k > 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{13}$$

2) If n is even and k is even:

$$H_{s,q}(\hat{\Pi}(S^n), Z) = \begin{cases} Z & \text{for } k = 0 \text{ and } s = n, \\ H_{s,q}(\hat{\Omega}(S^n), Z) & \text{for } (2k-1)(n-1) \leqslant s \leqslant 2k(n- \\ & -1) - 1,\ k > 0, \\ Z_4 & \text{for } s = 2k(n-1), \\ 0 & \text{for } s = 2k(n-1)+1, \\ H_{s-n,q}(\hat{\Omega}(S^n), Z) & \text{for } 2k(n-1)+2 \leqslant s \leqslant \\ & \leqslant (2k+1)(n-1)+1,\ k > 0, \\ 0 & \text{in the other cases.} \end{cases} \tag{14}$$

The proof is analogous to the proof of the corresponding theorems for the space of unoriented loops.

Remark 1. The cohomology ring of the space is computed analogously to (2).

Remark 2. By the same method one can compute the homologies of the spaces of oriented loops and closed paths (for all n).

4. Type numbers of closed geodesics

We give the corrected (in comparison with (1)) values of the type numbers of nondegenerate closed geodesics on an arbitrary Riemannian manifold. Let $i(g)$ denote the index of a closed geodesic g, let $m_0^s(g)$ be the s-dimensional type number of the geodesic g in the space of closed curves $K(M)$ or $\hat{K}(M)$ over the field of rational numbers, and let $m_p^s(g)$ be the type number modulo a prime p. Let g_k be a closed geodesic which is the k-fold iterate of a simple closed geodesic g_1.

Theorem 5. *The type numbers of a nondegenerate closed geodesic of multiplicity k on a Riemannian manifold M are equal to:*

1) If the multiplicity k is odd, or the multiplicity k is even and the number $i(g_2) - i(g_1)$ is also even:

$$m_0^s(g_k) = \delta^s_{i(g_k)},$$

$$m_p^s(g_k) = 1 \quad \text{for} \quad i(g_{d_p}) + 2 \le s \le i(g_k), \quad \text{where} \quad k = d_p \cdot p^l,\ (d_p, p) = 1, \tag{15}$$

$$m_p^s(g_k) = 0 \text{ in the other cases.}$$

2) If the multiplicity k is even, and the number $i(g_2) - i(g_1)$ is odd:

$$m_0^s(g_k) = 0,$$

$$m_p^s(g_k) = 0 \quad \text{for } p \neq 2,$$

$$m_2^s(g_k) = 1 \quad \text{for} \quad i(g_{d_2}) + 2 \leq s \leq i(g_k), \quad \text{where} \quad k = d_2 \cdot 2^l,\ d_2 \text{ odd}, \tag{16}$$

$$m_2^s(g_k) = 0 \text{ in the other cases.}$$

For the proof one considers an orthogonal periodic transformation in Hilbert space with a complete basis consisting of eigenfunctions of the linear differential Jacobi operator (with periodic boundary conditions). Note that for any k either $i(g_k) - i(g_1) \geq 1$, or $m_0^{i(g_k)}(g_k) = 0$, whence it follows that, if $m_0^s(g_1) \neq 0$, then for any k we have $m_0^{s+1}(g_k) = 0$ (cf. (4)).

Remark. The type numbers of nondegenerate extremal manifolds are computed analogously. In the particular case of the sphere S^n, the eigenfunctions are trigonometric functions or, in a discrete approximation, Chebyshev polynomials, which makes it possible to compute the indices and type numbers simply.

5. An estimate of the number of simple closed geodesics

Theorem 6. *Let M^n be a three-times continuously differentiable Riemannian manifold of positive curvature, diffeomorphic to the unit sphere S^n, and suppose that:*

1) there exists a diffeomorphism $v : M^n \to S^n$ under which the ratio of the Riemannian length of an arbitrary rectifiable curve g on M^n to the spherical length of this curve satisfies the inequality $1 \leq J(g)/S(g) < 2$;

2) the curvature at every point of the manifold M^n in every two-dimensional direction does not exceed 1 and is not less than $k_0 > 0$.

Put $n = 2^k + S$ $(S < 2^k)$. Then on the Riemannian manifold M^n there exist at least $2n - 1 - S$ simple closed geodesics.

The proof follows from (5) and V. A. Toponogov' s estimate (6). (For $n = 2q$, and also for $n = 2q + 1$ and $k_0 > 1/4$, the corresponding estimate was obtained by W. Klingenberg.)

Scientific Research Institute
of Applied Mathematics and Cybernetics
at Gorky State University
named after N. I. Lobachevsky

Received
12 VIII 1965

References

1. A. S. Schwarz, *Tr. Mosk. matem. obshch.*, **9**, 3 (1960).

2. S. I. Al' ber, *DAN*, **155**, No. 1 (1964).

3. S. I. Al' ber, *DAN*, **156**, No. 4 (1964).

4. A. I. Fet, *DAN*, **160**, No. 2 (1965).

5. S. I. Al' ber, *UMN*, **12**, 4 (76) (1957).

6. V. A. Toponogov, *DAN*, **154**, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.