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MAXIMAL AND NORMAL SERIES OF FINITE GROUPS

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§ 1. The main results of the present note concern the study of the influence of structural properties of the set of maximal subgroups of a group on its normal structure. Here, by structural properties of one or another set M of subgroups of a given finite group we mean those which are connected with the mutual relations between subgroups from M. These include, for example, the property of permutability of subgroups and the character of their intersections. The normal structure of a group, according to Wielandt, includes properties connected with the factors of normal series of the group. Such are, for example, the properties of solubility and generalized solubility.

In §§ 3 and 4 of the present note we study groups for which certain terms of their maximal series are quasi-subnormal in the group (see the definition below) or permutable. Problems of this kind were considered in papers ([1,6,10]), but there only invariant subgroups were used. The results given below show that subgroups of a more general kind (quasi-subnormal and permutable) in many cases exert the same influence on the normal structure of a group as invariant subgroups. This is confirmed, for example, by assertions 1) and 3) of Theorem 1, and by Theorems 3 and 4. In Theorem 2 one obtains the same class of nonsoluble groups as in ([10]), where invariant subgroups are considered.

In § 5 results are given on the influence of maximal subgroups having a given index or a given core on the solubility of a finite group. Theorem 8 generalizes a theorem of Ph. Hall ([16]), Theorem 10.5.7, while Theorem 9 generalizes a theorem of B. Huppert ([1]), Theorem 9). Particular cases of Theorems 8 and 9 are, respectively, Theorems 10 and 11 of the paper of A. V. Romanovskii ([15]).

In § 6 one new property of finite p-soluble groups is given. From Theorem 11, as consequences, one can obtain Theorem 1 of the paper of B. Huppert ([1]), as well as a theorem of G. Zappa ([13]) on the indices of maximal subgroups of strongly p-soluble groups.

§ 2. We shall give the notation and definitions used below.

G is a finite group of order (G); π is some (empty or not) set of primes; π' is the complement of the set π to the set of all primes; $\pi(G)$ is the set of all primes dividing (G); the notation $\pi\sigma\pi'$ means that for any $p \in \pi$ and $q \in \pi'$ one always has $p < q$; $\tau(G)$ is the number of elements of the set $\pi(G)$; $\tau_\pi(G)$ is the number of elements of π belonging to $\pi(G)$; a πd-group is a group whose order is divisible by some prime from π; E is the identity subgroup of the group G; $(G)_{\pi'}$ is the greatest π'-divisor ([5]) of the order of G; S_π^θ is a subgroup of the group G of order $(G)_{\pi'}$ possessing some group-theoretic property θ; $E_{\pi'}^\theta$ means that G has an S_π^θ-subgroup; group-theoretic properties: d is dispersiveness in the sense of Ore ([6]), z means that all Sylow subgroups of the group are cyclic; the core of a subgroup H is the intersection of all subgroups,

conjugate to H in G; a group of type S_1 is a nonnilpotent group all of whose proper subgroups are abelian (a Miller–Moreno group). The notions of p-solvability and strong p-solvability are taken by us from (2).

If $G \neq E$, then a series of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = E, \qquad n \geqslant 1,$$

is called a maximal series of the group G, if each member of this series G_i $(i = 1, 2, \dots, n)$ is a maximal subgroup of the preceding member G_{i-1}. If $G = E$, then the unique maximal series of the group G will be taken to be $E = E$.

By the i-th maximal subgroup of the group G $(i \geqslant 1)$ we shall mean the i-th member of some maximal series of it. Γ_i is the set of all i-th maximal subgroups of G; Γ_1^π is the set of all maximal πd-subgroups of G; Γ_2^π is the set of all second π-maximal subgroups (14) of the group G.

§ 3. A subgroup H of a group G is called quasireachable if, for any Sylow p-subgroup P of G (p ranges over the entire set $\pi(G)$), the intersection $H \cap P$ is a Sylow subgroup in H (see, for example, (7)). In particular, if (H) is not divisible by p, then the p-Sylow subgroup of H will be taken to be E.

It should be noted that the question remains open whether every quasireachable subgroup of some group G is reachable (subinvariant) in G. The converse assertion, as is not hard to verify, is true (8).

Theorem 1. *If in a group G: 1) all subgroups from the set Γ_1 are quasireachable, or if 2) all subgroups from the set Γ_2 are quasireachable, or if 3) all subgroups from the set Γ_3 are quasireachable, then the group G is, respectively: 1) nilpotent, 2) either nilpotent or a group of type S_1, 3) solvable, and for $\tau(G) > 3$ even nilpotent.*

Theorem 2. *If in a group G all subgroups from the set Γ_4 are quasireachable, then G will be of one of the following types:*

1. *A solvable group.*

2. $G \cong SL(2,5)$.
3. $G \cong LF(2,p)$, *where* $p = 5$, *or* p *is such a prime number that* $p-1$ *and* $p+1$ *are products of no more than three (not necessarily distinct) primes, and* $p \equiv \pm 3 \pmod{40}$ *or* $p \equiv \pm 13 \pmod{40}$.

Theorem 2 generalizes Theorems 1 and 2 of (10).

§ 4. Let U and V $(U \neq V)$ be certain sets of subgroups of the group G. Then the notation $UV = VU$ means that each subgroup from U permutes with each subgroup from V. We shall further say that the set U is weakly commutative if all subgroups from U that are conjugate to one another in the group G are pairwise permutable.

Theorem 3. *If in a* πd*-group* G *the set* Γ_1^π *is weakly commutative and* $\tau_\pi(G) > 1$, *then the group* G *is nilpotent.*

For $\pi(G) \subseteq \pi$, Theorem 3 yields a theorem of O. Ore ((6), Ch. II, § 2, Theorem 1).

Theorem 4. *Let* G *be a nonnilpotent* πd*-group and let the set* Γ_2^π *be weakly commutative. Then* G *is of one of the following types:*

1. *A group of type* S_1.
2. *The direct product of a cyclic group of order* p *by a group of type* S_1, *whose order is not divisible by* p.

From Theorem 4 one can derive Theorem 23 of (1) and Theorem 23 of (14). At the same time, Theorem 4 sharpens the assertion of Theorem 23 from (14).

Theorem 5. *If in a group* G *one of the following conditions is satisfied: 1)* $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, *2)* $\Gamma_1\Gamma_3 = \Gamma_3\Gamma_1$, *3)* $\Gamma_2\Gamma_3 = \Gamma_3\Gamma_2$, *then the group* G *is, respectively: 1) supersolvable, 2 and 3) solvable.*

The following definition was first introduced for consideration by S. A. Chunikhin in paper ([3]) (see also ([4])).

Definition. A subgroup H of a group G is called π-permutable in G if it is permutable with every p-Sylow subgroup of G for any $p \in \pi$.

Theorem 6. Let H be a proper πd-subgroup of the group G, with $H_G = E$, and let $\tau_\pi(G) > 1$. If H is π-permutable in G, then the group G contains, for each prime divisor $p \in \pi$ of the order of H, a normal divisor of index $p^\alpha n_1$, where $(p, n_1) = 1$ for any $p \in \pi$.

From Theorem 6 there follows one assertion of Theorem $1'$ of paper ([11]), establishing an analogous property for a quasinormal ([6]) subgroup, i.e. for a subgroup that is permutable with all subgroups of the given group. In addition, it follows from Theorem 6 that a simple πd-group for which $\tau_\pi(G) > 1$ cannot contain a proper π-permutable πd-subgroup.

Theorem 7. Let G be a solvable group and

$$G = G_1 G_2 \dots G_k,$$

where $G_1, G_2, \dots, G_k$ are pairwise permutable supersolvable groups whose orders are pairwise relatively prime. If every maximal subgroup of the group G_i is permutable with all G_j, $i, j = 1, 2, \dots, k$, then the group G is supersolvable.

From Theorem 7, using the theorem of H. Wielandt ([9]), we obtain

Corollary. Let

$$G = G_1 G_2 \dots G_k,$$

where $G_1, G_2, \dots, G_k$ are nilpotent groups satisfying all the remaining conditions of Theorem 7. Then the group G is supersolvable.

Theorem 7 and the corollary from it generalize the result of B. Huppert (([12]), p. 164).

§ 5. **Theorem 8.** Let the group G satisfy the following conditions:
1) $\pi\mathfrak{S}\pi'$, 2) $E^d_{\pi'}$, 3) all maximal subgroups of G containing some of its $S_{\pi'}$-subgroup H have as their index either a prime number or the square of a prime number. Then the group G is solvable.

Theorem 9. Let the group G satisfy the following conditions:
1) $\pi\mathfrak{S}\pi'$, 2) $E^z_{\pi'}$. The group G is then and only then supersolvable if all maximal subgroups of G containing some of its $S^z_{\pi'}$-subgroup H have prime index.

If $\pi(G) \subseteq \pi$, then Theorem 8 becomes Theorem 10.5.7 of P. Hall from ([16]), and Theorem 9 becomes Theorem 9 of B. Huppert from ([1]). If, however, $\pi(G)$ contains only one prime number from the set π', then from Theorems 8 and 9 we obtain, respectively, Theorems 10 and 11 of A. V. Romanovskii from ([15]).

Theorem 10. Let p be the greatest number in $\pi(G)$. If all maximal subgroups of the group G whose order is divisible by the greatest p-divisor of the order of G have core E, and if every maximal subgroup of G with core E is supersolvable, then the group G is solvable.

§ 6. **Definition 1.** If p^m is the highest power of the number p occurring as an index of all maximal chains of the group G, then $m = m_p(G)$ will be called the maximal p-rank of the group G.

Following B. Huppert ([1]), we introduce, for p-solvable groups, the following

Definition 2. Let $p^{\alpha_1}, p^{\alpha_2}, \dots, p^{\alpha_k}$ be all the indices of the chief chains of a p-solvable group G that are powers of the number p. Put

$$\operatorname{Max} \alpha_i = r_p(G),$$

and call $r_p(G)$ the chief p-rank of the group G.

Theorem 11. $r_p(G) = m_p(G)$.

An analogous dependence between chief and maximal chains in finite solvable groups was established earlier by B. Huppert ([1]).

§ 7. 1. The example of the tetrahedron group shows that the class of nonnilpotent groups covered by assertion 2 of Theorem 1 is broader than the class of groups all of whose second maximal subgroups are invariant (see ([1]), Theorem 23).

2. Using the example of the icosahedron group (with a corresponding choice of the set–

ness of π) one can verify the essentiality of conditions 1 and 3 in Theorem 8, as well as the essentiality of the condition of Theorem 10 requiring that every maximal subgroup of the group with core E be supersolvable.

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Note: Figure translations are in progress. See original paper for figures.