AN EXISTENCE THEOREM FOR THE SOLUTION OF THE PROBLEM OF SHORT-RANGE WEATHER FORECASTING

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Abstract

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AN EXISTENCE THEOREM FOR THE SOLUTION OF THE PROBLEM OF SHORT-RANGE WEATHER FORECASTING

In works ([1,2]) the method of fractional steps (the splitting method, the method of weak approximation) was used to study the correctness of the Cauchy problem for a linear system of differential equations in Banach space. The present note is devoted to proving the existence of a smooth solution of a mixed problem for one quasilinear system of equations. This problem is, in a certain sense (the problem is considered in a domain unbounded in x, y), a model one with respect to the problem used for short-range weather forecasting ([3]).

Problem A. It is required to determine functions u, v, H of the independent variables x, y, p, t, satisfying the system of equations

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-lv+\frac{\partial H}{\partial x}=0,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+lu+\frac{\partial H}{\partial y}=0,\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}-\frac{\partial}{\partial p}\frac{p^2}{\mu}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial H}{\partial p}=0\end{gathered} \tag{1}$$

for $t > 0,\ 0 < p_0 < p < P$, with boundary conditions

$$\begin{aligned}&\left(\tfrac{\partial}{\partial t}+u\tfrac{\partial}{\partial x}+v\tfrac{\partial}{\partial y}\right)\tfrac{\partial H}{\partial p}=0 &&\text{for } p=p_0,\\ &p\left(\tfrac{\partial}{\partial t}+u\tfrac{\partial}{\partial x}+v\tfrac{\partial}{\partial y}\right)\tfrac{\partial H}{\partial p}+\alpha\tfrac{\partial H}{\partial t}=0 &&\text{for } p=P\end{aligned} \tag{2}$$

and initial conditions

$$u(x,y,p,0)=u_0(x,y,p),\qquad v(x,y,p,0)=v_0(x,y,p),$$

$$H(x,y,p,0)=H_0(x,y,p). \tag{3}$$

Here l is a smooth function of x, y; μ is a smooth function of p, with $0 < \mu_0 \le \mu \le \mu_1$; $p_0, P, \alpha, \mu_0, \mu_1$ are constants.

Let $\overline{U}(t) = (u, v, H, \partial H/\partial p)$, $D_t = \{S_t \times (p_0, P)\}$; S_t is the section in the (x, y, t)-space by the plane $t = \mathrm{const}$ of the cone with base S_0 and generators inclined in the direction of increasing t at the angle

$$\gamma = \operatorname{arc\,tg}(\varkappa + 1),$$

where

$$\varkappa = C + V_0, \qquad V_0 = \max_S \sqrt{u_0^2 + v_0^2},$$

$$C = \max\{\sqrt{4\mu_1},\ \sqrt{2\mu(P)/\alpha}\}.$$

Let $W_2^{m,k_0}(D_t)$ be the set of functions having generalized derivatives belonging to $L_2(D_t)$ up to order m, where differentiation with respect to p does not exceed order k_0.

Definition. We define the norm $\mathbf{U}(t)$ in W_2^{m,k_0} as follows:

$$\|\mathbf{U}(t)\|^2_{W_2^{m,k_0}} = \int_{S_t} \Bigg\{ \sum_{m_1=0}^{m} \sum_{n+r=m_1} \Bigg(\int_{p_0}^{P} \Bigg[\Big(\frac{\partial^{m_1} u}{\partial x^n \partial y^r} \Big)^2 + \Big(\frac{\partial^{m_1} v}{\partial x^n \partial y^r} \Big)^2 + \frac{p^2}{\mu} \Big(\frac{\partial^{m_1+1} H}{\partial x^n \partial y^r \partial p} \Big)^2 \Bigg] dp$$

$$+ \frac{\alpha P}{\mu(P)} \Big(\frac{\partial^{m_1} H(P)}{\partial x^n \partial y^r} \Big)^2 \Bigg) + \sum_{m_1=1}^{m} \sum_{\substack{1 \le k \le k_0 \\ n+r+k=m_1}} \int_{p_0}^{P} \Bigg[\Big(\frac{\partial^{m_1} u}{\partial x^n \partial y^r \partial p^k} \Big)^2 + \Big(\frac{\partial^{m_1} v}{\partial x^n \partial y^r \partial p^k}$$

$$+ \Big(\frac{\partial^{m_1+1} H}{\partial x^n \partial y^r \partial p^{k+1}} \Big)^2 \Bigg] dp \Bigg\} \, dx \, dy.$$

Theorem. If $l \in W_2^m(S_0)$, $\mu \in W_2^{k_0}(p_0, P)$, $\mathbf{U}(0) \in W_2^{m,k_0}(D_0)$ for $m \ge 3$, $k_0 \ge 2$, then there exists a $\delta > 0$, depending on $\|\mathbf{U}(0)\|_{W_2^{m,k_0}}$, such that for $0 < t < \delta$ problem A has a solution $\mathbf{U}(t) \in W_2^{m,k_0}(D_t)$.

The uniqueness of the solution of problem A under the conditions of the theorem was proved in [4]. There it was also proved that a solution of problem A exists in the class of functions analytic in x, y, t and admitting majorants uniform in p. By the method of analytic approximation, in [5] the existence of a smooth solution of problem A was proved. In the present paper we prove the theorem by another method. This method was proposed in [3, 6, 8] for the numerical solution of the short-range weather-forecast problem.

The solution $\mathbf{U}(t)$ of problem A is constructed as the limit of solutions $\mathbf{U}_\tau(t)$ of problems A_τ as τ tends to zero. Problem A_τ "weakly approximates" problem A and consists in the successive solution of the mixed problem A' for system (1′) and the mixed problem A'' for system (1″).

Problem A_τ. The vector-function $\mathbf{U}_\tau(t)$ is continuous in t; $\mathbf{U}_\tau(0) = \mathbf{U}(0)$; for $2(n-1)\tau < t \leq (2n-1)\tau$ (n natural) it satisfies the system of equations

$$\begin{aligned} \frac{\partial u_\tau}{\partial t} + 2\frac{\partial H_\tau}{\partial x} &= 0, \\ \frac{\partial v_\tau}{\partial t} + 2\frac{\partial H_\tau}{\partial y} &= 0, \\ 2\left(\frac{\partial u_\tau}{\partial x} + \frac{\partial v_\tau}{\partial y}\right) - \frac{\partial}{\partial p}\frac{p^2}{\mu}\frac{\partial^2 H_\tau}{\partial p\,\partial t} &= 0 \end{aligned} \tag{1'}$$

and the boundary conditions

$$\begin{aligned} \frac{\partial^2 H_\tau}{\partial p\,\partial t} &= 0 \quad \text{for } p = p_0, \\ p\frac{\partial^2 H_\tau}{\partial p\,\partial t} + \alpha\frac{\partial H_\tau}{\partial t} &= 0 \quad \text{for } p = P; \end{aligned} \tag{2'}$$

for $(2n-1)\tau < t \leq 2n\tau$ it satisfies the system of equations

$$\begin{aligned} \frac{\partial u_\tau}{\partial t} + a_\tau\frac{\partial u_\tau}{\partial x} + b_\tau\frac{\partial u_\tau}{\partial y} - 2lv_\tau &= 0, \\ \frac{\partial v_\tau}{\partial t} + a_\tau\frac{\partial v_\tau}{\partial x} + b_\tau\frac{\partial v_\tau}{\partial y} + 2lu_\tau &= 0, \\ \frac{\partial^2 H_\tau}{\partial t\,\partial p} + a_\tau\frac{\partial^2 H_\tau}{\partial x\,\partial p} + b_\tau\frac{\partial^2 H_\tau}{\partial y\,\partial p} &= 0 \end{aligned} \tag{1''}$$

and the boundary condition

$$\frac{\partial H}{\partial t} = 0 \quad \text{for } p = P, \tag{2''}$$

where

$$a_\tau \equiv 2u_\tau((2n-1)\tau), \qquad b_\tau \equiv 2v_\tau((2n-1)\tau).$$

The solution of problem A_τ is reduced to the successive solution of two very simple problems A' and A'' (problem A', for example, can be solved by the Fourier method). Relying on the embedding theorem of S. L. Sobolev ([7]), one can show that both of these problems are uniformly well posed, in the sense of definition ([1]), in $W_2^{m,k_0}(D_t)$.

The reserve of smoothness ($m \geq 3,\ k_0 \geq 2$) ensures compactness of the set $\{U_\tau(t)\}$ for t less than some δ, determined by the initial data, and the existence of a limiting vector-function $U(t) \in W_2^{m,k_0}(D_t)$, which is the limit of $U_{\tau_k}(t)$ as $\tau_k \to 0$. If $U_\tau(t)$ is substituted into the equations of system (1) and the boundary conditions (2), then they will be satisfied with an error equal to right-hand sides that weakly approximate zero. Hence it follows that $U(t)$ is a solution of problem A, and, by virtue of the uniqueness of the solution of problem A, it follows that $U^\tau(t)$ converges to $U(t)$ as τ tends to zero.

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Note: Figure translations are in progress. See original paper for figures.