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MATHEMATICS

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A STRUCTURAL METHOD FOR DETERMINING DIMENSION

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This work is devoted to the structural determination of the dimension of a topological space, when the dimension is first defined for structures and even for arbitrary partially ordered sets, and then this dimension induces the structural dimension of a topological space. The structural dimension so defined for normal spaces with a countable base coincides with the usual one.

Let M be a partially ordered set. The exact least upper bound of a finite number of elements $a_1, a_2, \dots, a_n$, if it exists, will be called the **exact sum** of these elements, and we shall write

$$a = \bigcup_{i=1}^{n} a_i,$$

if for any element $b \subset a$ the intersections $b \cap a_i$ exist for all a_i, and

$$b = \sup b \cap a_i.$$

We shall call a set W of elements of M a **filter** if it satisfies the following requirements: 1) the intersection of a finite number of elements of W, if it exists, belongs to W; 2) if $a \in W$ and $b \supset a$, then $b \in W$; 3) if

$$\bigcup_{i=1}^{n} a_i \in W,$$

then at least one of the a_i belongs to W; 4) W does not contain the zero element (if this element exists). A filter that is contained in no other filter is called **maximal**. Every filter is contained in some maximal filter.

The idea of the definition of dimension is that filters of an arbitrary partially ordered set are interpreted by analogy with the filters of the following simple example.

Let M_n be the set of all bounded closed convex polyhedra of n-dimensional Euclidean space E, partially ordered by inclusion. Choose a point $x \in E$, then a ray r_1 having x as its endpoint, a two-dimensional half-plane r_2, whose boundary contains r_1, $\cdots$, an n-dimensional half-space r_n, whose boundary contains r_{n-1}. Denote by W_0 the set of polyhedra containing the point x; by W_1, the set of polyhedra belonging to W_0 and containing at least one point of r_1 not coinciding with x; $\cdots$; by W_n, the set of polyhedra belonging to W_{n-1} and containing at least one point of r_n not contained in r_{n-1}.

It is not hard to see that each of the sets W_i is a filter of M_n, and, conversely, every filter of M_n is a member of some decreasing sequence of filters

$$W_0 \supset W_1 \supset \cdots \supset W_n,$$

constructed in the manner described.

By analogy with this example we give the following definition.

Definition 1. A partially ordered set has **dimension** n if it contains a decreasing sequence

$$W_0 \supset W_1 \supset \cdots \supset W_n,$$

consisting of $n + 1$ filters, and there does not exist—

of a decreasing sequence consisting of a larger number of filters. We shall also give a general definition, when to each partially ordered set a dimension is assigned, equal to a finite or infinite cardinal number, in such a way that in the finite-dimensional case this definition coincides with the preceding one.

We shall call a set of filters $\{W\}$ a **dimensional system** if it satisfies the requirements: 1) if $W_1, W_2 \in \{W\}$, then either $W_1 \subset W_2$, or $W_2 \subset W_1$; 2) if $W_1, W_2, W_3 \in \{W\}$ and $W_1 \subset W_2$, $W_2 \subset W_3$, then $W_1 \subset W_3$.

Definition $1'$. The **dimension of a partially ordered set** M is the least upper bound of the set of cardinalities of all dimensional systems of M.

Above, examples were given of partially ordered sets of finite dimension. We note also that the set of bounded closed convex sets of n-dimensional Euclidean space has dimension n. Another example is supplied by the following theorem.

Theorem 1. *A distributive lattice with zero and unit has dimension zero if and only if it is a Boolean algebra.*

In the proof of this and of the other theorems the following lemma from (^{1}) is used.

Lemma. *Suppose that on a partially ordered set M there is given a nonnegative function $f(a)$ satisfying the conditions: 1) if $a \subset b$, then $f(a) \leq f(b)$; 2) if $a = \bigcup_{i=1}^{n} a_i$, then $f(a) \leq \sum_{i=1}^{n} f(a_i)$; 3) $f(0) = 0$.*

Then for every element $a \in M$ for which $f(a) > 0$, there exists a filter $W \ni a$, on all elements of which the function f is greater than zero.

Let M be a distributive lattice with zero and unit e of dimension zero. M is a Boolean algebra if every one of its elements has a complement. Suppose that there exists an element $a \in M$ having no complement. Define the function $f(b)$ to be equal to one if the element $a \cap b$ has no relative complement in b, and to zero otherwise. It is not difficult to show that this function satisfies the conditions of the lemma. Therefore, since $f(e) = 1$, there exists a filter W on whose elements the function f is greater than zero. Since $f(a) = 0$, we have $a \notin W$. At the same time all elements on which f is greater than zero, and hence all elements of W, have nonzero intersection with a. Therefore the maximal filter containing W must contain the element a and does not coincide with W, which contradicts the zero-dimensionality of the lattice M.

Conversely, suppose that a Boolean algebra has dimension greater than zero. Then there exist two filters W_0, W_1, for which $W_0 \supset W_1$. Take elements $a \in W_1$ and $b \in W_0 \setminus W_1$. Then also $a \cap b \in W_0 \setminus W_1$. But then $a - a \cap b \in W_1 \subset W_0$, which is impossible.

It is not difficult also to give examples of partially ordered sets of dimension τ for any cardinal number τ.

Let now E be a topological space. A lattice M, whose elements are the closed sets of the space E, is called **basic** if it contains the empty set and the whole space E, and if the complements of the sets from M form an open base of the space E.

Definition 2. The **structural dimension** of a topological space E is the least of the numbers τ for which E has a basic lattice of dimension τ; the structural dimension is denoted by $\operatorname{str} E$.

$\operatorname{str} E$ has the monotonicity property. Indeed, let E be a subspace of a topological space E', and let M' be a basic lattice of E'. Sets of the form $A \cap E,\ A \in M'$, form a basic lattice M of the space E. If W is a filter of M, then the collection W' of ⋯

sets $A \in M'$ for which $A \cap E \in W$, is a filter M'; moreover, if $W_1 \subset W_2$, $W_1 \neq W_2$, then also $W_1' \subset W_2'$, $W_1' \neq W_2'$. Therefore the dimension of M is less than or equal to the dimension of M', and $\operatorname{str} E \leq \operatorname{str} E'$.

Theorem 2. *The structural dimension of a topological space E is equal to zero if and only if the small inductive dimension* $\operatorname{ind} E = 0$.

If $\operatorname{str} E = 0$, then, by Theorem 1, there exists in E a basic structure which is a Boolean algebra. Therefore all sets of this structure are open-closed and $\operatorname{ind} E = 0$. Conversely, if $\operatorname{ind} E = 0$, then the structure of all open-closed sets of the space E is basic and at the same time is a Boolean algebra. Therefore $\operatorname{str} E = 0$.

Theorem 3. *If a topological space E has finite structural dimension, then E has finite inductive dimension and* $\operatorname{str} E \geq \operatorname{ind} E$.

If $\operatorname{str} E = 0$, then our assertion has been proved.

Let $\operatorname{str} E = n$, and suppose that the assertion of the theorem has been proved for all spaces whose structural dimension is less than n. Take in E a basic structure M of dimension n, and denote, for $A \in M$, by Γ_A the boundary of the set A, and by M_{Γ_A} the structure of sets of the form $B \cap \Gamma_A$, $B \in M$; M_{Γ_A} is a basic structure of the space Γ_A. We shall show that the dimension of the structures M_{Γ_A} is not greater than $n-1$. Suppose this is not so. Then there exists an $A \in M$ such that the structure M_{Γ_A} has a decreasing sequence of filters $W_0 \supset W_1 \supset \cdots \supset W_n$. Associate with each filter W_i the filter W^i of the structure M, consisting of all sets $B \in M$ for which $B \cap \Gamma_A \in W_i$.

Define on the structure M a function $f(B)$, equal to one if the closure $[B \setminus B \cap A]$ belongs to the filter W^n, and equal to zero otherwise. It is not difficult to verify that this function satisfies the conditions of the lemma given in the proof of Theorem 1. If $f(E) = 1$, then, by this lemma, there exists a filter W on which f is equal to one. Since $f(A) = 0$, it follows that $A \notin W$. At the same time $A \in W^n$ and $W \subset W^n$, $W \neq W^n$. Therefore there exists a decreasing sequence of filters of M, $W^0 \supset W^1 \supset \cdots \supset W^n \supset W$, consisting of $n+2$ terms, which is impossible, since the dimension of M is equal to n. If, however, $f(E) = 0$, then $\Gamma_A = [E \setminus A] \cap A \in W^n$, which is impossible. Therefore for any $A \in M$ the dimension of M_{Γ_A}, and hence also the structural dimension of Γ_A, is not greater than $n-1$, and, by the assumption made, $\operatorname{ind} \Gamma_A \leq n-1$. Since the complements of the sets of M form a base of E, it follows that $\operatorname{ind} E \leq n$. The question whether the inductive dimension of a topological space is not always equal to its structural dimension remains open. However, one can indicate a sufficiently general class of spaces for which such equality holds.

We give the following definitions from (2). A **directed set of closed coverings of a space** E is a set $\Sigma = \{\gamma\}$ of finite closed coverings of E which becomes a directed partially ordered set if an order relation is introduced in it as follows: a **covering** γ'' **follows the covering** γ' if each element of the covering γ' is the union of all elements of γ'' contained in it. A directed set $\Sigma = \{\gamma\}$ of closed coverings of the space E is called **refining** if for any point $x \in E$ and any neighborhood Ox of this point there is a covering $\gamma \in \Sigma$ such that the star of the point x in the covering γ is contained in Ox.

Theorem 4. *If in the space E there is a refining set of closed coverings of multiplicity $\leq n+1$, then* $\operatorname{str} E \leq n$.

Let $\Sigma = \{\gamma\}$ be a refining set of closed coverings of multiplicity $\leq n+1$ of the space E. Denote by M the smallest a structure of the set E, containing the empty set and all elements of all covers $\gamma \in \Sigma$. Since Σ is a refining set of covers, the complements to finite sums of elements $\gamma \in \Sigma$ form a base of the space E, and hence M is a basic structure. We shall show that the dimension of M is not greater than n. For a filter W of the structure M and for a cover $\gamma \in \Sigma$, denote by $\zeta_\gamma(W)$ the number of elements of the cover γ that are contained in W. Let $W_1 \subset W_2$ be two filters of M. Then for any cover $\gamma \in \Sigma$ there exists a cover $\gamma' > \gamma$ such that for all $\gamma'' \geq \gamma'$,

$$\zeta_{\gamma''}(W_1) \leq \zeta_{\gamma''}(W_2) - 1.$$

Indeed, if this is not so, then, since Σ is a directed set, for any $\gamma \in \Sigma$ there is a cover $\gamma' > \gamma$ for which $\zeta_{\gamma'}(W_1) = \zeta_{\gamma'}(W_2)$. Let F be an element of the cover γ belonging to W_2. Since F is the sum of the elements of γ' contained in it, F contains an element of γ' belonging to W_2. But all elements of γ' belonging to W_2 also belong to W_1, and therefore $F \in W_1$. It follows that $W_1 = W_2$. Since $\zeta_\gamma(W) \leq n+1$, no decreasing sequence of filters can contain more than $n+1$ terms, and the dimension of the structure M does not exceed n, whence $\operatorname{str} E \leq n$. I. V. Proskuryakov (^{3}) showed that in a normal space E with a countable base and with $\dim E \leq n$ there exists a refining set of closed covers of multiplicity $\leq n+1$. From Proskuryakov' s theorem and Theorems 3 and 4 it follows that

Theorem 5. For normal spaces with a countable base,

$$\dim E = \operatorname{str} E.$$

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Note: Figure translations are in progress. See original paper for figures.