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Abstract

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ON GAS FLOWS IN THE NEIGHBORHOOD OF A WEAK DISCONTINUITY

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Consider the problem of the adjoining of an isentropic two-dimensional unsteady flow of a polytropic gas to a region of gas at rest. The region of rest and the region of disturbed flow are separated by the surface of a weak discontinuity, on which the density and the components of the velocity vector u_i are continuous, while their derivatives suffer a discontinuity of the first kind. In the case when the flow behind the weak discontinuity is one-dimensional plane, it belongs to the class of simple waves. If, however, the flow is nonuniform and the surface of the weak discontinuity is curvilinear, then the flow behind the discontinuity does not belong to the class of simple waves. In the general case one can say about it only that it will be potential. The potential character of the flow follows at once from the kinematic compatibility conditions on the weak discontinuity.

In this note the adjoining of flows of the type of potential double waves (see ([1],[2])) to a region of rest is studied, and the application of these flows to the construction of an approximate picture of the motion in the neighborhood of certain arbitrary weak discontinuities.

p. 1. Assuming that outside the surface of the weak discontinuity u_1 and u_2 are functionally independent, we write the equations of double waves in polar coordinates r, φ, $u_1 = r\cos\varphi$, $u_2 = r\sin\varphi$, in the form

$$\frac{\gamma-1}{2}\theta\left[\theta_{rr}\left(1-\frac{\theta_\varphi^2}{r^2}\right)+\frac{1-\theta_r^2}{r^2}\theta_{\varphi\varphi}+2\frac{\theta_r\theta_\varphi}{r^2}\theta_{r\varphi}+\frac{\theta_r}{r}(1-\theta_r^2)-2\frac{\theta_r\theta_\varphi^2}{r^3}\right]+\frac{\gamma-3}{2}\left(\theta_r^2+\frac{\theta_\varphi^2}{r^2}\right)+2=0; \quad (1,1)$$

$$\Phi_{rr}\left(1-\frac{\theta_\varphi^2}{r^2}\right)+\frac{1-\theta_r^2}{r^2}\Phi_{\varphi\varphi}+2\frac{\theta_r\theta_\varphi}{r^2}\Phi_{r\varphi}+\frac{\Phi_r}{r}(1-\theta_r^2)-2\frac{\theta_r\theta_\varphi}{r^3}\Phi_\varphi=0, \quad (1,2)$$

$$x_1=\left[r\cos\varphi+\frac{\gamma-1}{2}\theta\left(\theta_r\cos\varphi-\theta_\varphi\frac{\sin\varphi}{r}\right)\right]t+\Phi_r\cos\varphi-\Phi_\varphi\frac{\sin\varphi}{r}, \quad (1,3)$$

$$x_2 = \left[r\sin\varphi + \frac{\gamma-1}{2}\theta\left(\theta_r\sin\varphi + \theta_\varphi\frac{\cos\varphi}{r}\right)\right]t + \Phi_r\sin\varphi + \Phi_\varphi\frac{\cos\varphi}{r}.$$

Here γ is the adiabatic exponent; $\theta = \dfrac{2}{\gamma-1}c$; c is the speed of sound; Φ is the velocity potential; the lower indices r and φ denote differentiation with respect to r and φ, respectively. The region of rest $u_1 = u_2 = 0$ corresponds, in the coordinates r, φ, to the segment of the axis $r = 0$. Equations (1,3) serve for the construction of the flow in the physical space x_1, x_2, t after the functions θ and Φ have been determined.

Consider the class of solutions of the equations of double waves for which, in a neighborhood of $r = 0$, $\Delta r = h$, $h < 1$, all fourth derivatives of the function θ containing differentiation twice with respect to r and twice with respect to φ are continuous. Let us note in passing that this class includes all corresponding self-similar cylindrical flows with self-similar variable $\sqrt{x_1^2 + x_2^2}/t$.

Estimating the terms in (1.1) in a neighborhood of $\Delta r = h$ and retaining only terms of order $O(1)$, we obtain for θ the equation

$$\frac{\gamma-1}{2}\theta\left[\theta_{rr} + \frac{\theta_r(1-\theta_r^2)}{r}\right] + \frac{\gamma-3}{2}\theta_r^2 + 2 = 0. \tag{1.4}$$

The equation for Φ takes the form

$$r^2\Phi_{rr} + (1-\theta_r^2)\Phi_{\varphi\varphi} + r(1-\theta_r^2)\Phi_r = 0. \tag{1.5}$$

Assuming that (1.3) at $r = 0$ determines the motion of some curvilinear weak discontinuity, for θ and Φ we obtain the initial data

$$\theta = \frac{2}{\gamma-1}, \qquad \theta_\varphi = 0, \qquad |\theta_r| = 1,$$

$$\Phi = 0, \qquad \Phi_\varphi = 0, \qquad \Phi_r = \Pi(\varphi) \quad \text{for } r = 0. \tag{1.6}$$

Here we have assumed that in the region of gas at rest $c = 1$, while in the perturbed region u_i are referred to the unperturbed sound speed, t is measured in seconds, and corresponding units have been chosen for x_i. The function $\Pi(\varphi)$, if at $t = 0$ the line of weak discontinuity is given by the equation $F(x_1, x_2) = 0$, is determined from the ordinary differential equation

$$F(\Pi\cos\varphi - \Pi'\sin\varphi,\ \Pi\sin\varphi + \Pi'\cos\varphi) = 0.$$

For r close to zero, from (1.4)–(1.6) we shall have, for θ and Φ,

$$\theta = \frac{2}{\gamma - 1} \pm r + \frac{\gamma + 1}{4} r^2 + C(\varphi) r^3; \tag{1.7}$$

$$r\Phi_{rr} + 2A\Phi_{\varphi\varphi} + 2rA\Phi_r = 0, \tag{1.8}$$

where $C(\varphi)$ is an arbitrary function, $A = \mp(\gamma + 1)/2$.

In the case when the gas density increases with distance from the surface of the weak discontinuity, $\theta_r = 1$ (for example, a one-dimensional weak discontinuity behind a normal detonation wave; see ([3])), $A < 0$, and equation (1.8) (and (1.5)) is of hyperbolic type for $r > 0$. Conversely, in the case of decreasing density (for example, when a weak discontinuity moves through gas at rest, producing a rarefaction wave), $\theta_r = -1$, and (1.8) (and (1.5)) is of elliptic type. In both cases the line $r = 0$, on which $\Phi_\varphi = 0$, is a line of parabolic degeneration for equation (1.8), while at the same time being a characteristic.

The conditions (1.6) do not in general determine the function θ for equation (1.4) uniquely. However, to accuracy $O(h^2)$, from (1.7) for θ we obtain a perfectly definite expression. The problem with data on the line of parabolicity is, in general, not well posed in either the hyperbolic or the elliptic case (see ([4])). However, in the hyperbolic case the problem with the initial data (1.6) for equation (1.8) (or for (1.5) with a fixed function θ), by theorem 1 of ([5]), has a unique twice continuously differentiable solution in the domain bounded by some segment of the axis $r = 0$ and the characteristics passing through its endpoints. This solution can be found by the Fourier method, since the variables in (1.8) and (1.5) separate. We note that equations (1.4), (1.5) have been used for other purposes in ([6]).

The question of the existence of a solution in the elliptic case with arbitrary $\Pi(\varphi)$ remains open. By a theorem of M. V. Keldysh (see ([7])), for equation (1.5) in a domain bounded by a segment of the axis $r = 0$ and a smooth curve Γ resting on this segment, there exists a solution of the Dirichlet problem. In this case the contour may be chosen arbitrarily; moreover, one may prescribe Φ arbitrarily on Γ (with $\Phi = 0$ for $r = 0$). Each such solution gives rise to a certain function $\Pi(\varphi)$ and, consequently, to a certain

surface of the weak discontinuity. With the help of a family of such solutions one can approximate the prescribed function $\Pi(\varphi)$. For this same purpose one may also use the Fourier method, applying regularization methods to the posed ill-conditioned problem.

Let us note that the system of equations (1,4), (1,5) determines a certain class of exact solutions of the equations of gas dynamics. Equation (1,4) can be obtained from the equations of cylindrical self-similar motion with independent variable $\sqrt{x_1^2 + x_2^2}/t$ (see (8)).

2. We shall show that, under certain conditions, the flow behind the surface of an arbitrary weak discontinuity can, in general, from a certain instant of time on, be approximately regarded as a double wave, and thus the apparatus of double waves provides an effective possibility of obtaining an approximate picture of the flows in some neighborhood of an arbitrary weak discontinuity of the type described.

Let the motion of the surface of the weak discontinuity be determined by the equation $t = \Psi(x_1, x_2)$, where Ψ satisfies the characteristic equation

$$c^2(\Psi_1^2 + \Psi_2^2) = 1, \qquad \Psi_i = \partial\Psi/\partial x_i. \tag{2,1}$$

Then along the bicharacteristic lying on this surface,

$$\dot{t} = 1, \qquad x_1 = c^2\Psi_1, \qquad x_2 = c^2\Psi_2, \tag{2,2}$$

the transport equation for the scalar σ, determining the jump in the derivatives of u_i and c emerging behind the surface $t = \Psi(x_1, x_2)$, will be

$$\dot{\sigma} + \frac{\gamma+1}{2c}\sigma^2 + \frac{c^2(\Psi_{11} + \Psi_{22})}{2}\sigma = 0, \qquad \Psi_{ik} = \frac{\partial^2\Psi}{\partial x_i \partial x_k}. \tag{2,3}$$

In (9) the transport equation for linear systems is derived, and the possibility of obtaining it for quasilinear systems is indicated.

In the present case we have a Riccati equation. Its general solution along a fixed bicharacteristic (2,2) can be written in the form

$$\sigma = \frac{1}{(\gamma+1)(t+B)} + \frac{1}{D(t+B)^{3/2} - (\gamma+1)(t+B)}, \tag{2,4}$$

where D is an arbitrary constant; B is a certain constant, its own for each bicharacteristic. In the class of double waves σ has the form

$$\sigma = 1/(\gamma+1)(t+B_1), \qquad B_1 = \text{const}. \tag{2,5}$$

We shall consider a neighborhood Δ_k of the weak discontinuity $t = \Psi(x_1, x_2)$, characterized by the fact that for $t \geq t_0$, for any point (x_1, x_2, t) belonging to this neighborhood, there is found on the surface $t = \Psi$ a point (x_1^0, x_2^0, t) such that

$$|x_1 - x_1^0| \leq k, \qquad |x_2 - x_2^0| \leq k. \tag{2,6}$$

Let the radius of curvature of the weak discontinuity increase with time, disturbances in the flow behind the discontinuity not overtake the discontinuity (the flow in a neighborhood of the discontinuity is sufficiently smooth), and, moreover, at the time $t = t_0 > |B_1|$ the scalar σ in (2,4), along a certain arc of the weak discontinuity, be determined by the values of the constants D, B, $d_0 \leq D \leq d_1$, $b_0 \leq B \leq b_1$; $d_0, d_1, b_0, b_1 = \text{const}$ in such a way that as t increases σ does not become infinite. Then from (2,4), (2,6) it follows that, in the case of an arbitrary weak discontinuity, in a neighborhood Δ_k, for sufficiently large t, $t_0 \sim O(k^{-2/3})$, with accuracy $O(k^2)$ $(k < 1)$, the solution may approximately be regarded as a double wave, and the problem is consequently reduced to solving equations (1,4), (1,5), or, in a cruder approximation, (1,8), with the initial conditions posed above.

What has been said makes it possible to obtain an approximate solution of a number of problems, for example, when weak discontinuities arise as curvilinear pistons of arbitrary shape are advanced, according to an arbitrary law, from a certain volume of gas at rest, and the flows are such that no perturbations from the flow region reach the weak discontinuity.

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Note: Figure translations are in progress. See original paper for figures.