ON A CLASS OF OPERATORS

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Abstract

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MATHEMATICS

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ON A CLASS OF OPERATORS

IN A SPACE WITH AN INDEFINITE METRIC

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Let $\mathfrak{H}$ be some J-space, i.e., a Hilbert space in which, along with the usual scalar product (x, y), an indefinite scalar product $[x, y] = (Jx, y)$ is given, where $J = \mathfrak{P}_+ - \mathfrak{P}_-$, $\mathfrak{P}_+$ and $\mathfrak{P}_-$ are orthoprojectors, $\mathfrak{P}_+ + \mathfrak{P}_- = I$.

We shall adhere, where not otherwise specified, to the terminology and notation of the paper (^{1}).

By $\mathfrak{B}$ we shall denote the ring of all linear bounded operators acting in $\mathfrak{H}$. For $A \in \mathfrak{B}$, by $\mathfrak{R}(A)$ will be denoted the set of all values of the operator A, by $\mathfrak{N}(A)$ the set of all its zeros, and by $\sigma(A)$ its spectrum. If $A \in \mathfrak{B}$, then A^+ denotes the J-adjoint operator to the operator A: $[Ax, y] = [x, A^+y]$ $(x, y \in \mathfrak{H})$. By $\mathfrak{H}_+$ $(\mathfrak{H}_-, \mathfrak{H}_0)$ we shall denote the set of all $x \in \mathfrak{H}$ for which $[x, x] \geq 0$ $(\leq 0, = 0)$.

Definition 1. An operator $A \in \mathfrak{B}$ will be called a **plus-operator** if $A\mathfrak{H}_+ \subset \mathfrak{H}_+$.

Definition 2. An operator $A \in \mathfrak{B}$ is called ***J*-noncontracting** (***J*-plus-noncontracting**) if $[Ax, Ax] \geq [x, x]$ for all $x \in \mathfrak{H}$ (for all $x \in \mathfrak{H}_+$).

The class of J-plus-noncontracting operators was first introduced, under another name, in 1950 in the paper (^{2}) (see also (^{3})) for J-spaces with $\min(\dim P_+\mathfrak{H}, \dim P_-\mathfrak{H}) < \infty$. The class of J-noncontracting operators was studied independently (^{4,5}). It is clear that, multiplying any J-noncontracting operator by a number λ, $|\lambda| \geq 1$, we obtain a certain J-plus-noncontracting operator. It turns out that in this way one can obtain all J-plus-noncontracting operators.

Still broader is the class of plus-operators. But here, too, it has turned out that under quite general conditions a plus-operator is collinear with a J-noncontracting operator. All these assertions are immediate consequences of the very general and completely elementary Theorem 1.

In the present note various properties of plus-operators are investigated. Special attention is paid to the question of when the operator A, together with its J-adjoint A^+, belongs to the class of plus-operators, and in connection with this the polar representation of plus-operators is studied. Elsewhere the authors set forth applications of the results obtained to the theory of fractional-linear transformations with operator coefficients.

1. In this section are collected propositions valid in any space $\mathfrak{L}$ over the field of complex numbers on which some sesquilinear Hermitian form $[x,y]$ is given, essentially indefinite (i.e., the functional $[x,x]$ assumes on $\mathfrak{L}$ values of different signs). As in the case of J-spaces, by $\mathfrak{L}_+$ ($\mathfrak{L}_-$) we shall denote the set of those $x \in \mathfrak{L}$ for which $[x,x] \geq 0$ (≤ 0).

Let $\Omega(x,y)$ be a second sesquilinear Hermitian form defined on $\mathfrak{L}$.

Theorem 1. If $\Omega(x,x) \geq 0$ when $[x,x]=0$, then:

1°.

$$\mu_\Omega = \inf_{[x,x]=1} \Omega(x,x) > -\infty.$$

2°. $\Omega(x,x) \geq \mu_\Omega [x,x]$ for all $x \in \mathfrak{L}$.

Under a stronger basic requirement ($\Omega(x,x) > 0$ when $[x,x]=0,\ x \neq \theta$), and also under some additional conditions concerning $\mathfrak{L}$ and the form $[x,y]$, the theorem was obtained by P. Kune (^{6}). At the same time, the proof of Theorem 1 required only the most insignificant changes in that author' s arguments.

Corollary 1^***.** If $\Omega(x,x)=0$ when $[x,x]=0$, then there exists a real number k such that $\Omega(x,y)=k[x,y]$ $(x,y \in \mathfrak{L})$.

Thus, a sesquilinear Hermitian form is uniquely determined, up to a scalar multiplier, by the set of its zeros on the "diagonal" $x=y$.

Corollary 2. Let A be a linear operator mapping $\mathfrak{L}$ into itself and satisfying the condition $A\mathfrak{P}_+ \subset \mathfrak{P}_+$. Then

$$[Ax,Ax] \geq \mu(A)[x,x] \qquad (x \in \mathfrak{L}), \tag{1}$$

where

$$\mu(A) = \inf_{[x,x]=1} [Ax,Ax] \geq 0.$$

Thus, if $\mu(A) > 0$, then $[Ax,Ax] > 0$ when $[x,x] > 0$; if $\mu(A)=0$, then $\mathfrak{M} = A\mathfrak{L}$ is a linear manifold consisting of nonnegative elements, i.e. $[y,y] \geq 0$ $(y \in \mathfrak{M})$.**

Corollary 3. Let A $(A\mathfrak{L} \subset \mathfrak{L})$ be annihilated only at zero and let $A\mathfrak{P}_+ = \mathfrak{P}_+$. Then there exists a number $k > 0$ such that

$$[Ax,Ay] = k[x,y] \qquad (x,y \in \mathfrak{L}). \tag{2}$$

Corollary 4^{***}**.** If $A\mathfrak{P}_\pm \subset \mathfrak{P}_\pm$, then there exists a number $k \geq 0$ such that equality (2) holds.

2. Let us return to the consideration of the case where $\mathfrak{L} = \mathfrak{H}$ is a J-space. Corollaries 2—4 of Theorem 1 are directly applicable to plus-operators. In particular, according to Corollary 2, to every plus-operator A there corresponds a number $\mu(A) \geq 0$ such that (1) holds.

Definition 3. A plus-operator A is called **strict** if $\mu(A) > 0$. It follows immediately from Definition 2 that an operator A is a strict plus-operator if and only if it is collinear to a J-noncontractive one.

Theorem 2. If A is a strict plus-operator, then there exists $\delta > 0$ such that $\|Ax\| \geq \delta\|x\|$ $(x \in \mathfrak{P}_+)$.

As δ one may take, for example,

$$\mu(A)/\left[\|A\| + \left(\|A\|^2 + \mu(A)\right)^{1/2}\right].$$

Corollary. A strict plus-operator maps every nonnegative (positive, uniformly positive) linear manifold homeomorphically onto a nonnegative (positive, uniformly positive) linear manifold.

If the operator A is J-noncontractive, it is not necessary that A^+ be J-noncontractive; moreover, the operator A^+ may even fail to be a plus-operator. The operator A is called doubly J-noncontractive if both it and A^+ are J-noncontractive operators.

* An equivalent assertion was obtained by H. Langer (private communication).

** These assertions should be compared with the results of M. L. Brodskii from his note ([7]).

*** In the works ([1,8]), continuously invertible operators A satisfying the condition $A\mathfrak{P}_\pm \subset \mathfrak{P}_\pm$ were considered in J-spaces. It follows from Corollary 4 that these operators are collinear to a J-unitary operator, and therefore Theorem 10 of ([1]) and Theorem 2 of ([8]) do not contain any new information in comparison with Corollary 11 and Theorem 3, respectively. M. G. Krein used this case to point out an annoying misprint that had crept into the formulations of the aforementioned theorems: instead of the condition $A\mathfrak{P}_\pm \subset \mathfrak{P}_\pm$, the condition $A\mathfrak{P}_+ \subset \mathfrak{P}_+$ is given. Corollary 4 should be compared with one result of Bognár ([9]).

Theorem 3. For a strict plus-operator A the following assertions are equivalent:

1°. A^+ is a strict plus-operator.

2°. A is collinear to a doubly J-noncontracting operator.

3°. A maps every maximal nonnegative subspace onto a maximal nonnegative subspace.

4°. A maps at least one maximal nonnegative subspace onto a maximal nonnegative subspace.

3. It turns out that certain properties of a plus-operator A determine one or another character of the spectrum of the J-Hermitian operator A^+A. This circumstance plays a fundamental role in deriving and analyzing the **polar representation** of a plus-operator (for the precise definition see below).

In finite-dimensional spaces the polar representation of J-noncontracting operators was exhaustively studied by V. P. Potapov (^{4}). The results of V. P. Potapov were extended by Yu. P. Ginzburg to the case of doubly J-noncontracting operators in infinite-dimensional J-spaces (^{5}). For this author, the polar representation of a doubly J-noncontracting operator A was obtained under one of two additional conditions: 1) $A^{-1} \in \mathfrak{B}$; 2) the operator $A - I$ is completely continuous. Before formulating a more general result (Theorem 5), we give the following proposition:

Theorem 4. If A is a plus-operator, then the spectrum $\sigma(A^+A)$ is real; if, moreover, A^+ is a plus-operator, then the spectrum $\sigma(A^+A)$ is nonnegative.

The first assertion follows from the fact that the operator $A^+A - \mu(A)I$ is J-nonnegative, and therefore has a real spectrum (^{5}). The second assertion contains, as a special case, the theorem of V. P. Potapov—Yu. P. Ginzburg:

If A is a doubly J-noncontracting operator, then the spectrum $\sigma(A^+A)$ is nonnegative.

In proving the second assertion, as well as the subsequent Theorems 5 and 6, one uses the spectral representation of J-nonnegative operators obtained by M. G. Krein and G. Langer on the basis of the methods of their work (^{10}) (set forth in detail in the doctoral dissertation of G. Langer, Dresden, 1964).

Definition 4. An operator $U \in \mathfrak{B}$ is called **partially J-isometric** (p. J-i.) if there exists a J-orthogonal decomposition of the space $\mathfrak{H}$: $\mathfrak{H} = \mathfrak{H}_1 \dot{+} \mathfrak{H}_2$, such that: 1) $[Ux, Ux] = [x, x]$ $(x \in \mathfrak{H}_1)$; 2) $Ux = \theta$ $(x \in \mathfrak{H}_2)$.

Definition 5. A **polar representation** of an operator $A \in \mathfrak{B}$ is a representation of it in the form $A = UR$, where U is a p. J-i. operator, and R is a J-Hermitian operator with nonnegative spectrum such that

$$R^2 = A^+A, \qquad \mathfrak{D}(R) = \mathfrak{D}(A^+A).$$

Theorem 5. In order that an operator A admit a polar representation $A = UR$ with J-noncontracting R, it is necessary and sufficient that three conditions be fulfilled:

1) the operator A is J-noncontracting; 2) the spectrum $\sigma(A^+A)$ is nonnegative and 3) the subspace $\mathfrak{D}(A^+)$ is regular*.

When these conditions are fulfilled, the following holds:

I. The operator R in any polar representation of the operator A is determined uniquely.

II. The operator U is determined uniquely if and only if at least one of the following three additional conditions is fulfilled: a) $\mathfrak{D}(A) = \{0\}$; b) $\mathfrak{D}(A^+) = \{0\}$; c) the subspace $\mathfrak{D}(A^+)$ is uniformly positive.

The necessity of conditions 1)–3) is proved comparatively simply. Let us dwell on certain points in the proof of the sufficiency of these conditions. In deriving the polar representation of the operator A, a fundamental role is played by the possibility of extracting the square root of the operator A^+A. Condition 2) of the theorem and the existence of a spectral representation for the operator A^+A make it possible to extract this root, i.e. to find a J-self-adjoint operator R with nonnegative spectrum such that $R^2 = A^+A$, $\mathfrak{Z}(R) = \mathfrak{Z}(A^+A)$.

As for assertion I, it can be established by special arguments in a strengthened form. It turns out that the operator $R \in \mathfrak{B}$ is determined uniquely by two requirements: 1) $R^2 = A^+A$ and 2) the spectrum $\sigma(R)$ is nonnegative.

The equality $R^2 = A^+A$ is equivalent to the condition

$$[Rx, Rx] = [Ax, Ax] \qquad (x \in \mathfrak{H}). \tag{3}$$

Define on $\mathfrak{R}(R)$ an operator V by putting $VRx = Ax$ $(x \in \mathfrak{H})$. It is not difficult to verify that condition 3) ensures the correctness of this definition. By construction, the operator V is J-isometric on $\mathfrak{R}(R)$. Using the theorem proved by I. S. Iokhvidov on the boundedness of a J-isometric operator with regular domains of definition and ranges (11,12), one can prove the boundedness of the operators V and V^{-1}. After this it remains to extend the closure $\overline{V}$ of the operator V in some way to a p.u. operator U.

Obviously, the operator U is determined uniquely if and only if $\overline{V}$ admits no proper J-isometric extensions in $\mathfrak{H}$. The latter holds when one of the conditions a), b), c) is satisfied, and only in this case.

Remark. If conditions 1)–3) are satisfied, it is not difficult to indicate additional requirements under which the operator U can be chosen to be J-unitary, J-semiunitary, or J-adjoint to a J-semiunitary one. For example, the first possibility occurs if and only if the subspace $\mathfrak{Z}(A^+)$ is uniformly negative and $\dim \mathfrak{Z}(A) = \dim \mathfrak{Z}(A^+)$.

It follows from Theorem 2 that for a J-noncontracting operator A, the subspace $\mathfrak{Z}(A)$ is either equal to $\{0\}$ or is negative. This fact was first observed by Yu. P. Ginzburg (5), who showed, moreover, that $\mathfrak{Z}(A)$ is uniformly negative (see (1), problem 7 on p. 39). Hence, from the Potapov–Ginzburg theorem cited above there follows the "necessary" part of the following proposition:

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* A subspace $\mathfrak{M} \subset \mathfrak{H}$ is called **regular** if $\mathfrak{H}$ is representable as the direct J-orthogonal sum of $\mathfrak{M}$ and its J-orthogonal complement.

Theorem 6. *In order that a J-noncontracting operator A be doubly J-noncontracting, it is necessary and sufficient that: 1) the spectrum $\sigma(A^+A)$ be nonnegative and 2) the subspace $\mathfrak{Z}(A^+)$ be uniformly negative.*

We prove sufficiency. The operator A satisfies the conditions of Theorem 5 and therefore admits a polar representation $A = UR$. Then $A^+ = RU^+$, $[A^+x, A^+x] = [RU^+x, RU^+x]$. From equality (3) it follows that the operator R is J-noncontracting, whence $[A^+x, A^+x] \geq [U^+x, U^+x]$. From condition 2) it is not difficult to derive that $[U^+x, U^+x] \geq [x, x]$ for all $x \in \mathfrak{H}$, so that $[A^+x, A^+x] \geq [x, x]$ $(x \in \mathfrak{H})$. The theorem is proved.

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