ON SOME PROPERTIES OF ALGEBRAS OF CONTINUOUS FUNCTIONS

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MATHEMATICS

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ON SOME PROPERTIES OF ALGEBRAS OF CONTINUOUS FUNCTIONS

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§ 1. **Basic definitions.** In this note the following notation and terms will be used: X is a compact Hausdorff space; $C(X)$ is the algebra of all complex continuous functions on X with the uniform norm; $M(X)$ is the space of all finite complex Borel measures on X; a measure is an element of the space $M(X)$; a partition is a covering of the space X by pairwise disjoint closed sets; $A \subset C(X)$ $-A$ is a closed subalgebra of the algebra $C(X)$ containing the constants; $f|F$ is the restriction of the function $f \in C(X)$ to the set F; $A|F$ is the algebra of all restrictions to F of functions from $A \subset C(X)$; $(A|F)^-$ is the uniform closure of the algebra $A|F$; $\operatorname{Re} f$ is the real part of the function f; $\operatorname{Re} A = \{\operatorname{Re} f : f \in A\}$; χ_F is the characteristic function of the set F.

If $A \subset C(X)$ and $F \subset X$, then F is called a set of antisymmetry (for A) if every real-valued function in $A|F$ is constant; F is a peak set (for A) if there exists a function $f \in A$ such that $f|F = 1$ and $|f(x)| < 1$ for $x \in X \setminus F$. A closed set $F \subset X$ will be called a set of weak analyticity (for A) if every peak set for the algebra $(A|F)^- \subset C(F)$ either coincides with F or is nowhere dense in F.

If $A \subset C(X)$ and $\mu \in M(X)$, then $\mu \perp A$ means that $\int f \, d\mu = 0$ for all $f \in A$.

§ 2. **The spaces H^∞.** Let $A \subset C(X)$. For every positive measure $\mu \in M(X)$, denote by $H^1(\mu)$ the closure of the algebra A in the space $L^1(\mu)$, and by $H^\infty(\mu)$ the intersection $H^1(\mu) \cap L^\infty(\mu)$. Put

$$\operatorname{Re} H^\infty(\mu) = \{\operatorname{Re} f : f \in H^\infty(\mu)\}.$$

Lemma 1. (1) $H^\infty(\mu)$ is a closed subalgebra of the algebra $L^\infty(\mu)$; (2) from every bounded sequence $f_n \in H^\infty(\mu)$ in $L^\infty(\mu)$ one can extract a subsequence converging weakly in $L^1(\mu)$ to some function $f \in H^\infty(\mu)$; (3) if $F \subset X$ and $\chi_F \in \operatorname{Re} H^\infty(\mu)$, then $\chi_F \in H^\infty(\mu)$.

We shall call a positive measure μ antisymmetric if every real-valued function in $H^\infty(\mu)$ is constant almost everywhere with respect to μ.

Lemma 2. (1) If μ is an antisymmetric measure, then the support $\operatorname{Sp}\mu$ of the measure μ is a set of weak analyticity; (2) every set of weak analyticity is a set of antisymmetry.

Lemma 3. If $f \in C(X)$ and $f|\operatorname{Sp}\mu \in A|\operatorname{Sp}\mu$ for every antisymmetric measure μ, then $f \in A$.

§ 3. **Restoring partitions.** Let $A \subset C(X)$. A partition $\mathcal{K}$ of the space X will be called restoring if: (1) from the conditions $f \in C(X)$ and $f|K \in A|K$ for each $K \in \mathcal{K}$ it follows that $f \in A$; (2) for each maximal ideal I of the algebra A there exists a unique $K \in \mathcal{K}$ for which $(I|K)^{-}$ is a maximal ideal of the algebra $(A|K)^{-}$.

Example 1. The Shilov partition $\mathcal{K}_0$. Points $x_1, x_2 \in X$ belong to one and the same $K \in \mathcal{K}_0$ if and only if $f(x_1) = f(x_2)$ for every real-valued function $f \in A$ (see (1), § 44).

Example 2. The Bishop partition $\mathcal{K}_1$. Its elements are the maximal, with respect to inclusion, antisymmetry sets (see ([2,3])).

The partitions $\mathcal{K}_0$ and $\mathcal{K}_1$ are recovering and, moreover, satisfy the condition: for every $K \in \mathcal{K}_i$ $(i = 0, 1)$ the algebra $A|K$ is closed in $C(K)$. Here we shall show that there exist recovering partitions finer than the Bishop partition.

Lemma 4. *Let $\mathcal{R}$ be some collection of subsets of the space X. Among all partitions $\mathcal{K}$ satisfying the condition: every $R \in \mathcal{R}$ is contained in some $K \in \mathcal{K}$, there exists a finest one. We shall denote this partition by $\mathcal{K}(\mathcal{R})$.*

Theorem 1. A. *Let $A \subset C(X)$. Then: 1) every set of weak analyticity is contained in a maximal one, and the collection $\mathcal{R}$ of all maximal sets of weak analyticity covers X; (2) if $f \in C(X)$ and $f|R \in A|R$ for all $R \in \mathcal{R}$, then $f \in A$; (3) the partition $\mathcal{K}_2 = \mathcal{K}(\mathcal{R})$ is recovering; 4) every element of the partition $\mathcal{K}_2$ is contained in some element of the Bishop partition $\mathcal{K}_1$.*

B. *If X is a metrizable space and $A \subset C(X)$, then for every $K \in \mathcal{K}_2$ the algebra $A|K$ is closed in $C(K)$.*

C. *If $A_1 \subset C(X)$, $A_2 \subset C(X)$ and the partitions $\mathcal{K}_2$ coincide for the algebras A_1 and A_2, then the partitions $\mathcal{K}_1$ also coincide for them.*

Proof. Assertion A(1) is verified with the aid of Zorn' s lemma; A(2) follows from Lemmas 2(1) and 3. Assertions A(3) and B, as well as Lemma 3, are proved by the methods developed in ([3]) (see pp. 416-417, 419, 421-422). Assertion A(4) follows from Lemma 2(2). To prove assertion C, note that if f is a real-valued function from A_1, then it is constant on every $K \in \mathcal{K}_2$ and, consequently, belongs to A_2. Therefore the partition $\mathcal{K}_0$ is uniquely determined by the partition $\mathcal{K}_2$. Applying the same reasoning to every element of the partition $\mathcal{K}_0$ and continuing thus by transfinite induction, we construct the partition $\mathcal{K}_1$.

We give an example showing that the partition $\mathcal{K}_2$ may be substantially finer than the partition $\mathcal{K}_1$.

Example. Let X_1 be the prism $0 \le x \le 1$, $0 \le t \le 1$, $|y| \le t$ in three-dimensional arithmetic space with coordinates x, y, t; let A_1 be the algebra of all functions defined and continuous on X_1 and analytic in the interior of each section $t = \text{const}$. Denote by X the compactum obtained by identifying the points $(s, 0, 0) \in X_1$ with the points $(0, 0, s) \in X_1$ for all $s \in (0, 1]$. Let φ be the natural projection of X_1 onto X, and let

$$A = \{f : \ f \in C(X); \ f \circ \varphi \in A_1\}$$

($f \circ \varphi$ denotes the superposition of the mapping φ and the function f). The algebra A separates the points of X, is closed in $C(X)$, and is antisymmetric. At the same time the partition $\mathscr{K}_2$ for the algebra A is nontrivial and consists of the layers $t = \text{const}$ $(0 < t \le 1)$ and the point $(0, 0, 0)$. Moreover, X is the space of maximal ideals of the algebra A.

§ 4. **The space** $\operatorname{Re} A$**.** In this paragraph we establish the connection between the partitions $\mathscr{K}_0$ and $\mathscr{K}_1$ and the space $\operatorname{Re} A$ of real parts of functions from the algebra A. First note (see (^{4})) that $\operatorname{Re} A$ may be considered as a real Banach space with norm N:

$$N(u) = \inf\{\|u + iv\| : \ u + iv \in A\} \quad (u \in \operatorname{Re} A).$$

Denote by C_R the set of all continuous real-valued functions defined on the real line.

Theorem 2. *Let* $u \in \operatorname{Re} A$*. In order that* $u \in A$*, it is necessary and sufficient that* $f \circ u \in \operatorname{Re} A$ *for all* $f \in C_R$*, where* $f \circ u$ *denotes the function defined on* X*:*

$$(f \circ u)(x) = f(u(x)).$$

Proof. Necessity follows directly from the Stone–Weierstrass theorem.

Let $f \circ u \in \operatorname{Re} A$ for all $f \in C_R$, and let μ be an antisymmetric measure (see § 2) and F the support of the measure μ. We shall show that $u|F = \text{const}$. If this is not so, then there is a bounded sequence of functions $p_n = f_n \circ u$ $(f_n \in C_R)$ in $C(X)$, converging pointwise to the characteristic function χ_G of some closed set $G \subset X$, whose intersection with F is different from F and has interior points in F. Clearly,

$$0 < \mu(G) < \mu(F).$$

But on the subspace $S = \{f \circ u : \ f \in C_R\}$ of the space $\operatorname{Re} A$, the norm N, as is easily checked, is equivalent to the uniform norm on X. Therefore the sequence p_n is bounded in the norm N, and there exists a sequence $g_n \in A$, bounded in $C(X)$, such that $\operatorname{Re} g_n = p_n$. Applying Lemma 1 (2) to the sequence g_n and then Lemma 1 (3), we obtain that

$$\chi_G \in H^\infty(\mu).$$

But this is impossible, since μ is an antisymmetric measure.

Thus the function u is constant on the supports of all antisymmetric measures. Since the algebra A contains the constants, by Lemma 3 we conclude that $u \in A$.

Corollary. If $A_1 \subset C(X)$, $A_2 \subset C(X)$, and $\operatorname{Re} A_1 = \operatorname{Re} A_2$, then the decompositions $\mathfrak{K}_0$ and $\mathfrak{K}_1$ for the algebras A_1 and A_2 coincide.

Glicksberg (^{3}) proved that if a closed G_δ-set F is such that the condition $\mu \perp A$ ($\mu \in M(X)$) implies $\chi_F \mu \perp A$, then F is a peak set for A. Using this result and arguing in essentially the same way as in the proof of Theorem 2, one can prove that the peak sets are determined by the space $\operatorname{Re} A$.

Theorem 3. Let $A_1 \subset C(X)$ and $A_2 \subset C(X)$. If $\operatorname{Re} A_1 = \operatorname{Re} A_2$, then every peak set for A_1 is a peak set for A_2.

We shall state the last theorem without proof. It relates the space $\operatorname{Re} A$ to the Gleason parts (^{5}) and to the decomposition $\mathfrak{K}_0$. Let $A \subset C(X)$. For arbitrary $x, y \in X$, put

$$\rho_A(x,y) = \sup\{|f(x) - f(y)| : \ f \in A, \ \|f\| = 1\}.$$

As Gleason showed, any two sets of the form

$$V_x = \{y : \ y \in X, \ \rho_A(x,y) < 2\}$$

(such sets are called parts) either coincide or are disjoint. For each set $F \subset X$, the number

$$D_A(F) = \sup\{\rho_A(x,y) : \ x, y \in F\}$$

will be called the diameter of the set F.

Now let $A_1 \subset C(X)$ and $A_2 \subset C(X)$. Denote by $(\operatorname{Re} A_1)^-$ the uniform closure of the space $\operatorname{Re} A_1$, and by $\mathfrak{K}_0(A_2)$ the Shilov decomposition for the algebra A_2.

Theorem 4. Let $A_1 \subset A_2$. In order that $(\operatorname{Re} A_1)^- \subset \operatorname{Re} A_2$, it is necessary and sufficient that there exist a number c, $0 \le c < 2$, such that

$$D_{A_1}(K) \le c$$

for every $K \in \mathfrak{K}_0(A_2)$.

Corollary 1. If A_2 is an antisymmetric algebra and $(\operatorname{Re} A_1)^- \subset \operatorname{Re} A_2$, then A_1 consists only of constants (this follows from the fact that $D_{A_1}(X)$ can be equal only to either 0 or 2).

Corollary 2 (Hoffman-Wermer theorem (^{6})). If an algebra $A \subset C(X)$ separates points and $\operatorname{Re} A$ is closed in the topology of uniform convergence, then $A = C(X)$.

Indeed, put $A_1 = A_2 = A$ in Theorem 4. Then every $K \in \mathfrak{K}_0(A_2)$ consists of a single point (otherwise $D_{A_1}(K) = 2$). Therefore $A = C(X)$.

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Note: Figure translations are in progress. See original paper for figures.

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