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Abstract

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ON MEASURES IN LINEAR SPACES

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Let X be a real linear space and let X' be some linear system of linear functionals on X. A natural number n, a collection of n functionals $f_1, \dots, f_n$ from X', and a Borel set A in the real n-dimensional space R^n determine in X the cylindrical set

$$C = C(f_1, \dots, f_n, A) = \{x \in X : [(f_1, x), \dots, (f_n, x)] \in A\} \tag{1}$$

with base A and generators $f_1, \dots, f_n$.

We shall say that a **premeasure** is given on the totality $\mathfrak{A}$ of all cylindrical sets if to each set $C \in \mathfrak{A}$ there is assigned a nonnegative number μC in such a way that: a) μC depends only on the set C itself and not on its representation in the form (1); b) on the totality $\mathfrak{A}_f$ of all cylindrical sets $C \in \mathfrak{A}$ with fixed collection $f = (f_1, \dots, f_n)$, the function μC is countably additive (countable additivity on all of $\mathfrak{A}$ is not assumed); c) $\mu X = 1$.

We pose the question under what conditions a premeasure μ can be extended to a measure—a countably additive set function defined on a σ-ring of subsets $E \subset X$ containing all cylindrical sets.

If X is the space of **all** real functions $x = x(t)$ on some set T, and the functionals $f \in X'$ are generated by the values of x at individual points $t \in T$, then a positive answer is the content of a theorem of A. N. Kolmogorov ([1]). There are a number of conditions due to various authors that clarify this question under certain assumptions concerning the existence and properties of a topology on X ([2,3]). The following simple result pertains to spaces without topology. Suppose that X' is a total system of functionals on X, i.e., from the equalities $(f, x) = 0$, satisfied for a fixed $x \in X$ for all $f \in X'$, it follows that $x = 0$. Then X can be regarded as a space of real functions $x(f) = (f, x)$, where f ranges over X' or even only over some total subset T in X'. Let $\hat{X}$ be the space of **all** real functions $x = x(f)$, $f \in T$, with the premeasure of cylindrical sets (with functionals f, equal to the values of the functions x at the points $f \in T$) given by the formula

$$\hat{\mu}\{x \in \hat{X} : [x(f_1), \dots, x(f_n)] \in A\} = \mu\{x \in X : [(f_1, x), \dots, (f_n, x)] \in A\}.$$

By Kolmogorov' s theorem, the premeasure $\hat{\mu}$ is completed in $\hat{X}$ to a countably additive measure defined on some σ-ring of subsets of the set $\hat{X}$.

Theorem 1. *The premeasure μ is completed in X to a measure if and only if the image of X under the mapping $x \to x(f) \in \hat{X}$ fills, in the space $\hat{X}$, a set of outer $\hat{\mu}$-measure* 1.

A premeasure μ is called **Gaussian** if to each collection of functionals $f = (f_1, \dots, f_n)$ there correspond a positive quadratic form $Q_f(\xi, \xi)$, $\xi \in R^n$, and a constant c_f such that, for any Borel set $A \subset R^n$,

$$\mu C(f_1, \dots, f_n, A) = c_f \int_A \exp[-Q_f(\xi, \xi)]\, d\xi_1 \cdots d\xi_n.$$

If, for the space X with Gaussian premeasure μ, a countable subsystem $g_1, g_2, \dots$ has been selected in X', then, applying the orthogonalization process to it, one can pass to a new countable subsystem $e_1, e_2, \dots$ in X' (with the same linear hull), for which

$$\mu C(e_1, \dots, e_n, A) = \frac{1}{\sqrt{\pi^n}} \int_A \exp\left[-\sum_1^n \xi_k^2\right] d\xi_1 \dots d\xi_n.$$

If the system $g_1, g_2, \dots$ is total, then the system $e_1, e_2, \dots$ is also total; in this case the space $\hat{X}$ is the space Ω of all real sequences $x = (x_1, x_2, \dots)$ with the canonical Gaussian measure

$$\omega\{x \in \Omega : (x_1, \dots, x_n) \in A\} = \frac{1}{\sqrt{\pi^n}} \int_A \exp\left[-\sum_1^n \xi_k^2\right] d\xi_1 \dots d\xi_n.$$

By virtue of Theorem 1, the Gaussian premeasure μ extends on X to a measure if and only if the image of X under the mapping

$$x \to [(e_1, x), (e_2, x), \dots] \in \Omega$$

fills in Ω a set of outer measure 1. Therefore, for applications it is essential to have a sufficiently broad supply of subsets of Ω of full measure. We give several examples (the original formulation of which was given earlier in the language of probability theory):

Example 1 (B. V. Gnedenko ([4])).

$$\omega\left\{x \in \Omega : \lim_{n\to\infty} \frac{|x_n|}{\sqrt{\ln n}} = 1\right\} = 1.$$

Example 2 (A. N. Kolmogorov ([5])). Let a function $g(\xi)$, $-\infty < \xi < \infty$, be given such that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(\xi) e^{-\xi^2}\, d\xi = a, \qquad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g^2(\xi) e^{-\xi^2}\, d\xi < \infty.$$

Then

$$\omega\left\{x \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} g(x_k) = a\right\} = 1.$$

Example 3 (A. N. Kolmogorov and A. Ya. Khinchin ([6])). Let positive numbers $a_1, a_2, \ldots$ be given; set

$$E_{\langle a_n\rangle} = \left\{x \in \Omega : \sum_{1}^{\infty} a_n x_n^2 < \infty\right\}.$$

Then

$$\omega E_{\langle a_n\rangle} = \begin{cases} 1, & \text{if } \sum_{1}^{\infty} a_n < \infty, \\ 0, & \text{if } \sum_{1}^{\infty} a_n = \infty. \end{cases}$$

Let $f_1, f_2, \ldots$ be a numerical sequence. By Kolmogorov' s theorem ([7]), the series

$$[f, x] \equiv \sum_{1}^{\infty} f_n x_n$$

converges on Ω in the mean square

(and also almost everywhere) if and only if $\sum_1^\infty f_n^2 < \infty$. In this case

$$\omega\{x \in \Omega : [f, x] > c\} = \frac{1}{\sqrt{\pi}} \int_{c/\left(\sum_1^\infty f_n^2\right)^{1/2}}^{\infty} e^{-\xi^2}\, d\xi.$$

The expression $[f, x]$ is the general form of a linear measurable functional on Ω ([8]). If several sequences $f_1^j, f_2^j, \ldots$ $(j = 1, \ldots, m)$, orthogonal and normalized in the space l_2, are given, then

$$\omega\{x \in \Omega : ([f^1, x], \dots, [f^m, x]) \in A\} = \frac{1}{\sqrt{\pi^m}} \int_A \exp\left[-\sum_1^m \xi_k^2\right] d\xi_1 \dots d\xi_m.$$

It follows that a linear transformation of the space Ω into itself, given by the formulas

$$y_n = \sum_{m=1}^{\infty} u_{mn} x_m$$

with an orthogonal matrix $U = \|u_{mn}\|$, carries measurable sets $E \subset \Omega$ again into measurable sets of the same measure. Fan Dyk Tinh showed in ([8]) that the indicated transformation is the general form of a measurable linear transformation in Ω preserving measure. There he also indicated the general form of linear transformations $y = Bx$ that carry measurable sets into measurable sets and do not preserve measure, but have an absolutely continuous set function $\omega(BE)$.

Wiener measure. Let H be a separable Hilbert space and $e_1, e_2, \dots$ an orthonormal basis in H. Each vector $z \in H$ is represented by a sequence of numbers $z_n = (e_n, z)$ with $\sum_1^\infty z_n^2 < \infty$. Fix a sequence of positive numbers $\lambda_1, \lambda_2, \dots$ and associate with the point $z \in H$ the point $(\lambda_1 z_1, \lambda_2 z_2, \dots)$ of the space Ω. Introduce in H the premeasure

$$\mu\{z \in H : (z_1, \dots, z_n) \in A \subset R_n\} = \omega\{x \in \Omega : (\lambda_1 z_1, \dots, \lambda_{nz} n) \in A\}.$$

The premeasure μ is extended in H to a measure if the image of H in Ω fills a set of full measure (Theorem 1). By the Kolmogorov-Khinchin criterion ([3]), this condition is satisfied if and only if the series $\sum_1^\infty \lambda_n^{-2}$ converges.

The measure in the space H obtained for $\lambda_n \equiv n$ will be called the **abstract Wiener measure**. If as H one takes the space $L_2(0, \pi)$ of all square-integrable functions $z(t)$ on the interval $[0, \pi]$, orthogonal to 1, and as basis vectors e_n takes the functions

$$e_n(t) = \sqrt{\frac{2}{\pi}} \cos nt,$$

then we obtain the **classical Wiener measure**. The functions $z(t) \in \bar{L}_2(0, \pi)$ that are continuous and satisfy the Hölder condition $|z(t') - z(t'')| \le C|t' - t''|^\alpha$ with any exponent $\alpha < 1/2$ fill in $\bar{L}_2(0, \pi)$ a set W of full measure. If from the functions $z(t) \in W$ we pass to the functions $\varphi(t) = z(t) - z(0)$, equal to 0 for $t = 0$, we obtain the second classical realization of Wiener measure ([9]). Kolmogorov' s criterion 2, after a certain modification, leads to a set of full measure (indicated by Cameron and Martin ([10]) for the special case $g(\xi) = \xi^2$):

$$\omega\left\{\varphi(t):\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}g\left[\frac{\varphi\left(\frac{k}{n}\pi\right)-\varphi\left(\frac{k-1}{n}\pi\right)}{\sqrt{\pi/n}}\right]=a\right\}=1.$$

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Note: Figure translations are in progress. See original paper for figures.