ON SPACES DECOMPOSABLE INTO SPECTRA CONSISTING OF METRIC SPACES

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ON SPACES DECOMPOSABLE INTO SPECTRA CONSISTING OF METRIC SPACES

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In the present paper we consider spaces that are limits of inverse spectra of metric spaces. Such spaces were considered earlier by B. A. Pasynkov in his work ([1]), where, in particular, necessary and sufficient conditions are given for the spectral decomposability of spaces with respect to the class of metric spaces. Theorem 1 establishes a one-to-one correspondence between the completion of a uniform space with respect to its given structure Σ and an inverse spectrum of complete metric spaces with uniformly continuous projections, called extremal. Next, the connection between spaces complete in the sense of Dieudonné and spectra of metric spaces is considered without using the concept of uniformity. Theorem 3 establishes, for finite-dimensional uniform spaces, the existence of spectra consisting of metric spaces of the same dimension.

By a spectrum here we shall always mean a directed set of spaces X_α, connected with one another by continuous mappings—projections $\mathfrak{D}_\alpha^\beta : X_\beta \to X_\alpha$, defined whenever $\beta > \alpha$, and satisfying the transitivity condition: $\mathfrak{D}_\alpha^\gamma = \mathfrak{D}_\alpha^\beta \mathfrak{D}_\beta^\gamma$ for $\gamma > \beta > \alpha$.

B. A. Pasynkov in ([2]) introduced the notion of an extremal spectrum for bicompacta and established a one-to-one correspondence between bicompacta and their extremal spectra. We define the notion of an extremal spectrum consisting of metric spaces.

Definition 1. Two spaces X_α and X_β of a spectrum S will be called S-homeomorphic if there exists such a homeomorphism $X_\beta \to X_\alpha$ that, under $x_\beta \to x_\alpha$, the conditions $\mathfrak{D}_\alpha^\gamma x_\gamma = x_\alpha$ and $\mathfrak{D}_\beta^\gamma x_\gamma = x_\beta$ are equivalent for every $\gamma > \alpha, \beta$. If this homeomorphism is uniform in both directions, then these two spaces will be called S-isomorphic.

Definition 2. A subspectrum S' of a spectrum S is called *splitting* if, for any X_{α_0} from S and for any two points x'_{α_0} and x''_{α_0} in X_{α_0}, there exists such a $\beta > \alpha_0$ that, for any two points x'_β and x''_β satisfying the conditions $\mathfrak{D}_{\alpha_0}^\beta x'_\beta = x'_{\alpha_0}$, $\mathfrak{D}_{\alpha_0}^\beta x''_\beta = x''_{\alpha_0}$, there is in S' a space X_γ in which $\mathfrak{D}_\gamma^\beta x'_\beta \neq \mathfrak{D}_\gamma^\beta x''_\beta$.

Definition 3. We shall say that the *order in the spectrum*

$$S = \{X_\alpha, \mathfrak{D}_\alpha^\beta\}$$

is *maximal* if there exists no spectrum $S' = \{X_\gamma, \mathfrak{D}^\gamma\}$, distinct from it, consisting of the same elements X_α, and such that from $\beta > \alpha$ in S' it follows that $\beta > \alpha$ in S.

Definition 4. A spectrum

$$S = \{X_\alpha, \mathfrak{D}_\alpha^\beta\}$$

is called *extremal* if it satisfies the following conditions: 1) it contains no S-homeomorphic spaces; 2) it has maximal order; 3) it is not

a splitting subspectrum of any spectrum satisfying conditions 1) and 2).

Definition 5. We shall call a spectrum $S = \{X_\alpha,\ \mathfrak{D}_\alpha^\beta\}$ **uniform** if all projections $\mathfrak{D}_\alpha^\beta$ are uniformly continuous.

Definition 6. A uniform spectrum $S = \{X_\alpha,\ \mathfrak{D}_\alpha^\beta\}$ is called **extremal** if it satisfies the following conditions: 1) it contains no S-isomorphic spaces; 2) it has a maximal order; 3) it is not a splitting subspectrum of any uniform spectrum satisfying conditions 1) and 2).

Definition 7. Two spectra $S = \{X_\alpha,\ \mathfrak{D}_\alpha^\beta\}$ and $S' = \{X_\gamma,\ \mathfrak{D}_\gamma^\nu\}$ will be called **equivalent** (**isomorphic**) if there exists an isomorphism between the sets of their indices and, for each pair of corresponding indices, a homeomorphism (a homeomorphism uniform in both directions) $X_\alpha \leftrightarrow X_\gamma$.

Theorem 1. *The completion of a uniform space X with respect to its structure Σ is the limit of a unique, up to isomorphism, uniform extremal spectrum $S = \{X_\alpha,\ \mathfrak{D}_\alpha^\beta\}$ of complete metric spaces, and the structure Σ is determined by the preimages of all metric covers of the elements of the spectrum S. Therefore there is a one-to-one correspondence between complete uniform spaces and their extremal spectra.*

Proof. Consider all possible uniformly continuous mappings $\mathfrak{D}_\alpha : X \to X_\alpha$ of the space X onto metric spaces. Take the completions $\widetilde{X}_\alpha$ of all these metric spaces X_α. We obtain a system of pairs $(\mathfrak{D}_\alpha, \widetilde{X}_\alpha)$, where $\widetilde{X}_\alpha$ are complete metric spaces and $\widetilde{\mathfrak{D}}_\alpha$ are uniformly continuous mappings. We shall regard the pairs $(\mathfrak{D}_\xi, \widetilde{X}_\xi)$ and $(\mathfrak{D}_\eta, \widetilde{X}_\eta)$ as isomorphic if there exists a homeomorphism, uniform in both directions, $\widetilde{X}_\xi \to \widetilde{X}_\eta$ such that $\mathfrak{D}_\xi = \mathfrak{D}_\xi^\eta \mathfrak{D}_\eta$. Divide the whole system into classes of S-isomorphic pairs and choose one representative from each class. The system of pairs thus obtained is directed; for any two pairs $(\mathfrak{D}_\xi, \widetilde{X}_\xi)$ and $(\mathfrak{D}_\eta, \widetilde{X}_\eta)$ one can take the pair $(\mathfrak{D}_\zeta, \widetilde{X}_\zeta)$, where

$$\mathfrak{D}_\zeta(x) = (\mathfrak{D}_\xi(x),\ \mathfrak{D}_\eta(x)) \in \widetilde{X}_\xi \times \widetilde{X}_\eta,$$

$$X_\zeta = \mathfrak{D}_\zeta(X) \subset \widetilde{X}_\xi \times \widetilde{X}_\eta,$$

$\widetilde{X}_\zeta$ is the completion of X_ζ, and $\widetilde{\mathfrak{D}}_\zeta$ is the extension of $\mathfrak{D}_\zeta$. We introduce an order: put $\beta > \alpha$ if there exists a uniformly continuous projection $\mathfrak{D}_\alpha^\beta$ satisfying the condition $\mathfrak{D}_\alpha = \mathfrak{D}_\alpha^\beta \mathfrak{D}_\beta$. Since in the system of pairs there are no S-isomorphic ones, it cannot happen simultaneously that $\beta > \alpha$ and $\alpha > \beta$. From the system of pairs obtained we construct a spectrum $S = \{X_\alpha,\ \mathfrak{D}_\alpha^\beta\}$. The limit of this spectrum will be the completion $\widetilde{X}$ of the space X with respect to the structure Σ. Indeed, first, for every α, $X_\alpha = \mathfrak{D}_\alpha(X)$; consequently X is everywhere dense in the limit of the spectrum S. Secondly, the structure Σ consists of the preimages of all metric covers of the spaces X_α. Our uniform spectrum S is extremal. The fulfillment of conditions 1) and 2) of Definition 6 follows from the very construction of the spectrum S. Suppose that condition 3) of this definition is not fulfilled, i.e. the spectrum S is a splitting subspectrum of another uniform spectrum S^*, not coinciding with it, of complete metric spaces. But then there exists a space X_{α^*} belonging to S^* and not belonging to S. Since the spectra S and S^* have one and the same limit, S^* also consists of the completions of all uniformly continuous images of $\widetilde{X}$ and, by the supposition, contains S^*-isomorphic spaces, which contradicts condition 1).

Let us now verify the uniqueness of the extremal spectrum. Suppose there is a uniform extremal spectrum S^*, consisting of complete metric spaces and having as its limit the space $\widetilde{X}$. We shall show—

namely, that it is a subspectrum of the spectrum S. Indeed, all mappings $\tilde{\omega}_\nu : \tilde{X} \to \tilde{X}_\nu \in S^*$ are uniformly continuous with respect to the structure Σ, induced by the metric coverings of the elements of the spectrum S^*. Hence all elements of the spectrum S^* are completions of uniformly continuous images of X. Since among all pairs $(\tilde{\omega}, \tilde{X}_\nu)$ from S^* there are no S^*-isomorphic ones, the spectrum S^* is a subspectrum of the spectrum S. Moreover, the spectra S and S^* have one and the same limit. Therefore S^* is a refining subspectrum of the spectrum S and, consequently, coincides with it.

Theorem 2. *Every complete uniform space in the sense of Diedonné is the limit of a unique, up to equivalence, extremal spectrum of metric spaces.*

Proof. Consider all possible pairs $(\tilde{\omega}_\alpha, X_\alpha)$, where X_α are metric spaces, and $\tilde{\omega}_\alpha$ are continuous mappings $\tilde{\omega}_\alpha : X \to X_\alpha$ onto these metric spaces. We introduce an order in the following way: $\beta > \alpha$, if there exists a mapping $\tilde{\omega}_\alpha^\beta : X_\beta \to X_\alpha$ satisfying the condition $\tilde{\omega}_\alpha = \tilde{\omega}_\alpha^\beta \tilde{\omega}_\beta$. Repeating the arguments of B. A. Pasynkov, carried out in (1) in constructing a completion of a space with the maximal structure, we obtain a spectrum $S = \{X_\alpha, \tilde{\omega}_\alpha^\beta\}$, whose limit is the space X and which satisfies conditions 1) and 2) of Definition 4. The proof of the fact that S is not a refining subspectrum of any other spectrum S^* satisfying conditions 1) and 2) is carried out exactly as in the preceding theorem: assuming the contrary, there exists in S^* a space X_{α^*} not belonging to S, and since S contains all pairs $(\tilde{\omega}_\alpha, X_\alpha)$, there must exist S^*-homeomorphic spaces in S^*. Uniqueness is proved by an almost literal repetition of the arguments of Theorem 1.

It is also not difficult to establish a one-to-one correspondence, up to equivalence, between complete spaces in the sense of Diedonné and extremal spectra of complete metric spaces.

Let us now consider finite-dimensional uniform spaces. We shall say, following Isbell, that a uniform space X has uniform dimension $\Delta dX = n$, if for every covering ω from its structure Σ there exists an $(n+1)$-fold covering $\eta \in \Sigma$ inscribed in ω.

The metric dimension μdM of a metric space M is the least number n such that for every $\varepsilon > 0$ there exists an open covering of the space M of multiplicity $\leq n+1$, each element of which has diameter less than ε.

Theorem 3. *If a complete uniform space X has uniform dimension $\Delta dX \leq n$, then in the extremal spectrum $S = \{X_\alpha, \tilde{\omega}_\alpha^\beta\}$ of the space X there is a cofinal subspectrum $S' = \{X_{\alpha'}, \tilde{\omega}_{\alpha'}^\beta\}$ consisting of metric spaces having metric dimension $\mu dX_{\alpha'} \leq n$.*

Proof. Take an arbitrary covering ω_0 from the given structure Σ of the space X. Since $\Delta dX \leq n$, there exists a normal sequence $\{\omega_i\}$ of coverings of multiplicity $\leq n+1$, inscribed in ω_0 and belonging to the structure Σ. Take the metric space X_{ω_0} induced by this sequence. We shall show that $\mu dX_{\omega_0} \leq n$. Indeed, let $\varepsilon = 1/2^k$ be given. Consider ω_{k+3}. We shall assume that the set of elements of the covering ω_{k+2} is well ordered, $\omega_{k+2} = \{U_\lambda\}_{\lambda\in\Lambda}$. Denote by $O_{\omega_{k+3}}A$ the star of the set A with respect to the covering ω_{k+3}, $O_{\omega_{k+3}}A = \bigcup\{U;\ U \cap A \neq \Lambda,\ U \in \omega_{k+3}\}$. The system of open sets $\{f(O_{\omega_{k+3}}x)\}_{x\in X}$ is a $1/2^{k+1}$-covering of X_{ω_0}. Put

$$O_\lambda = \bigcup\{O_{\omega_{k+3}}x;\ O_{\omega_{k+3}}x \subseteq U_\lambda\}.$$

The system $\{O_\lambda\}_{\lambda\in\Lambda}$ is a covering of the space X of multiplicity not exceeding $n+1$. For each λ the set $f(O_\lambda)$ is open in X_{ω_0}, and its diameter is less than ε. Thus, for every covering $\omega \in \Sigma$ there exists a uniform ω-mapping onto a metric space X_ω of dimension $\mu dX_\omega \leq n$. Take the system of pairs $(\mathfrak{F}_\alpha, X_\alpha)$, where the X_α are metric spaces, $\mu dX_\alpha \leq n$, and the $\mathfrak{F}_\alpha$ are uniformly continuous mappings. Setting $\beta' > \alpha'$ whenever there exists a uniformly continuous mapping $\mathfrak{F}_{\alpha'}^{\beta'} : X_{\beta'} \to X_{\alpha'}$, we obtain a spectrum $S' = \{X_{\alpha'}, \mathfrak{F}_{\alpha'}^{\beta'}\}$ over a directed set, which is a cofinal subspectrum of the extremal spectrum S.

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Note: Figure translations are in progress. See original paper for figures.