ON FIELDS ASSOCIATED WITH ENVELOPING ALGEBRAS OF LIE ALGEBRAS

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Abstract

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In many questions of analysis, noncommutative rings arise naturally. Examples include rings of operators in Hilbert space, differential operators on a smooth manifold, operators of quantum field theory, group rings of groups, and enveloping algebras of Lie algebras. One of the most general and interesting examples is the ring of all differential operators on a smooth manifold X that are invariant with respect to a given group G of diffeomorphisms of X. The study of the algebraic structure of this ring is essential for analysis.

In the present note we restrict ourselves only to the special case of such rings —enveloping algebras of Lie algebras. As in algebraic geometry, the situation here is simplified if one adopts the birational point of view and, instead of rings, considers fields of fractions. Let us pass to precise definitions.

Let G be a Lie algebra defined over a field L of characteristic 0, and let $U(G)$ be the enveloping algebra of the algebra G. It can be shown that $U(G)$ is a (two-sided) Ore domain, i.e., a ring without zero divisors in which any two elements have a common (right and left) multiple. Therefore one can define the field of fractions of the algebra $U(G)$, each element of which can be written in the form ab^{-1} and in the form $c^{-1}d$, where $a, b, c, d \in U(G)$. We shall denote this field by $D(G)$ and call it the Lie field of the algebra G. The center of $D(G)$ will be denoted by $C(G)$.

Let now K be an arbitrary field over the field L. Define the dimension of K over L by the following formula:

$$\operatorname{Dim}_L K = \sup_{\alpha} \inf_{b\neq 0} \lim_{N\to\infty} \frac{\ln d(\alpha b, N)}{\ln N}, \tag{1}$$

where $\alpha = (a_1, a_2, \dots, a_s)$ is an arbitrary finite set of elements of K; $\alpha b = (a_1 b, a_2 b, \dots, a_{sb})$; $d(\alpha, N)$ is the dimension of the subspace in K consisting of all elements that can be written as polynomials of degree $\leq N$ (with coefficients in L) in the elements of the set α. In the case when the field K is commutative, the notion of dimension introduced by us coincides, as is easy to verify, with the degree of transcendence.

Theorem 1. *Let A be an algebra over a field L of characteristic 0. Suppose that in A there exists a filtration*

$$L = A_0 \subset A_1 \subset \dots \subset A_k \subset \dots; \qquad A_{kA}l \subset A_{k+l},$$

such that the associated graded algebra

$$\operatorname{gr} A = \sum A_k/A_{k-1}$$

is isomorphic to the algebra of all polynomials in n variables (the grading in which need not coincide with the standard one).

Then A is an Ore domain and the field of fractions K of the algebra A has dimension n.

The proof of this theorem uses the following property of the field K. Let p_i be the natural projection of A_i onto $A_i/A_{i-1} \subset \operatorname{gr} A$. For every nonzero element $a \in A$ there exists a unique number i for which $p_i(a) \neq 0$. We shall call $p_i(a)$ the *leading term* of the element a and denote it by $[a]$.

Every element $b \in K$ can be written in the form $a_1 a_2^{-1}$, $a_1, a_2 \in A$. It turns out that the rational function $[a_1]/[a_2]$ depends only on the element b, and not on its representation in the form $a_1 a_2^{-1}$. We shall denote this function by $[b]$.

Lemma 1. *The mapping $b \mapsto [b]$ is a homomorphism of the multiplicative group of the field K into the group of rational functions of the form PQ^{-1}, where P, Q are homogeneous polynomials (in general, of different degrees) in n variables.*

This lemma is also useful in the study of automorphisms of the field K.

Let now G be an algebraic Lie algebra. Then the following holds.

Theorem 2. *The numbers n and k, defined by the equalities*

$$2n + k = \operatorname{Dim}_L D(G), \qquad k = \operatorname{Dim}_L C(G), \tag{2}$$

are nonnegative integers. Moreover, the equalities

$$2n + k = \dim G, \qquad k = \operatorname{codim} O, \tag{3}$$

hold, where O is an orbit in general position in the representation, dual to the adjoint one, of the algebraic group corresponding to the Lie algebra G.

Our main hypothesis consists in the assertion that, for algebraic Lie algebras, the numbers n and k determine the field $D(G)$ up to isomorphism.

Denote by $D_{n,k}(L)$ the field generated over the field L by the elements $x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n, y_1, \dots, y_k$ with the natural relations:

$$x_i y_j - y_{jx} i = 0, \qquad y_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} y_i = 0, \qquad \frac{\partial}{\partial x_j} x_i - x_i \frac{\partial}{\partial x_j} = \delta_{ij}.$$

The field $D_{n,k}(L)$ plays the role of a standard model. The center of $D_{n,k}(L)$, as is easily verified, coincides with the subfield $D_{0,k}(L) \subset D_{n,k}(L)$. From Theorem 1 it follows that

$$\mathrm{Dim}_L\, D_{n,k}(L) = 2n + k.$$

The main hypothesis mentioned above can be refined as follows.

(T). The field $D(G)$ is isomorphic to $D_{n,k}(L)$, where the numbers n and k are determined by the equalities (2) or (3).

Theorem 3. *Hypothesis (T) is valid in the following cases: 1) G is an arbitrary nilpotent Lie algebra; 2) G is the full matrix Lie algebra or the Lie algebra of matrices with trace zero; 3) G is a semisimple Lie algebra of rank 2 over an algebraically closed field.*

The proof of Theorem 3 is carried out by entirely different methods in each of these cases. We shall indicate the main stages of the proof in each case.

1) Hypothesis (T) is replaced by the following assertion. In the algebra $U(G)$ there exist elements $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_k$, which generate the field $D(G)$ and satisfy the relations

$$x_i x_j - x_{jx} i = y_i y_j - y_{jy} i = z_i z_j - z_{jz} i = x_i z_j - z_{jx} i = y_i z_j - z_i y_j = 0;$$

$$x_i y_j - y_{jx} i = \delta_{ij} C,$$

where C is a nonzero element of the center of $U(G)$.

This assertion is proved by induction on the dimension of the algebra G. In doing so, one uses certain results of Dixmier ([1]) and of one of the authors ([2]) concerning the relation between $U(G)$ and $U(G_0)$, where G_0 is an ideal of codimension 1 in G.

2) We consider the auxiliary Lie algebra G_n, isomorphic to the algebra of matrices of order $n+1$ whose last row consists of zeros. The validity of hypothesis (T) for matrix algebras G follows from results of one of the authors on the structure of the center of the algebra $U(G)$ ([3]) and from the validity of the hypothesis for G_n. The latter assertion is proved by induction on n on the basis of the following lemma.

Lemma 2. Let e_{ik}, $1 \le i \le n$, $1 \le k \le n+1$, be the natural basis in G_n. Denote by D' the subfield of $D(G_n)$ consisting of all elements that commute with e_{ii} and $e_{i,n+1}$, $1 \le i \le n$. Then the field D' is isomorphic to $D(G_{n-1})$.

It follows from this that

$$D(G_n) \simeq D_{n(n+1)/2,\,0}(L).$$

3) This is proved by the direct construction of a basis in $D(G)$ having the required properties. The construction is facilitated by the fact that the field $D(G)$ is generated by the subfield spanned by a maximal solvable

subalgebra M in G, and by the subfield consisting of elements commuting with the elements of M.

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Note: Figure translations are in progress. See original paper for figures.