ON SOME CLASSES OF FUNCTIONS ANALYTIC IN MULTIPLE HARTOGS DOMAINS

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ON SOME CLASSES OF FUNCTIONS ANALYTIC IN MULTIPLE HARTOGS DOMAINS

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In the present article, some results of the works ([1], [2]), established for spaces of functions analytic in bounded multiple Hartogs domains, are extended to certain classes of functions analytic in unbounded multiple Hartogs domains. To shorten the notation we shall use the designations $w = (z_1, \dots, z_m)$, $z = (z_{m+1}, \dots, z_n)$, $k = (k_1, \dots, k_m)$, $\|k\| = k_1 + \cdots + k_m$, $w^k = z_1^{k_1} \cdots z_m^{k_m}$.

§ 1. Let $D\{|z_j| < R_j(z),\ j = 1, \dots, m;\ z \in H_D\}$ be a complete m-fold Hartogs domain with planes of symmetry $z_1 = 0, \dots, z_m = 0$ over the space C^n (see ([1], [2])), whose projection H_D is unbounded; $A(D)$ is the space of functions $f(w, z)$, analytic in the domain D, with the topology determined by uniform convergence of functions from $A(D)$ in each domain lying strictly inside D. Consider the class $A_1(D) \subset A(D)$ of functions $f(w, z) \in A(D)$ bounded in D.

It is known ([1], [2]) that every function $f(w, z) \in A(D)$ is representable in the domain D by an m-fold Hartogs series

$$f(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k. \tag{1}$$

Theorem 1. If the function $f(w, z) \in A_1(D)$ and satisfies in D the condition $|f(w, z)| \le M$, then

$$|f_k(z)| \le \frac{M}{d_k(D)}, \qquad d_k(D) = \sup_{(w,z)\in D} |w^k|.$$

Theorem 2. If the series (1) converges in the domain D and $f(w, z) \in A_1(D)$, then for every subdomain D_0 of it, $\overline{D}_0 \subset D$, the series

$$\sum_{k=0}^{\infty} \sup_{D_0} |f_k(z)|\, d_k(D_0)$$

converges.

Corollary. If the series (1) converges in the domain D and $f(w,z) \in A_1(D)$, then in every subdomain D_0 of it, $\overline{D}_0 \subset D$, the series

$$\sum_{k=0}^{\infty} f_k(z) d_k(D_0)$$

converges uniformly.

Theorem 3. For convergence of the series (1) to the function $f(w,z) \in A_1(D)$ in the domain D, it is necessary and sufficient that the series

$$\sum_{k=0}^{\infty} \sup_{D} |f_k(z)|\, d_k(D) w^k$$

converge in the domain $D^{(1)}\{|z_j| < 1,\ j = 1, \dots, m;\ z \in H_D\}$.

Theorem 4. For convergence of the series (1) to the function $f(w,z) \in A_1(D)$ in the domain D, it is necessary and sufficient that the series

$$\sum_{k=0}^{\infty} f_k(z) d_k(D) w^k$$

converge in the domain $D^{(1)}$.

Theorems 1-4 are proved analogously to Theorems 1-4 of ([1]).

Analogously to ([1]), introduce into consideration the countably normed space $B_1(D)$ of sequences of functions $a = \{f_k(z)\}$ that are Hartogs coefficients of functions $f(w,z) \in A_1(D)$, with the system of norms

$$\|a\|_r = \sum_{k=0}^{\infty} \sup_{z \in H_D} |f_k(z)|\, d_k(D)\, r^{\|k\|}, \qquad 0 < r < 1.$$

We shall also consider the class $A_1(\overline{D}) \subset A(\overline{D})$ of bounded functions analytic in the closed domain $\overline{D}$, and the countably normed space $B_1(\overline{D})$ of sequences of their Hartogs coefficients $a = \{f_k(z)\}$, with the system of norms taken analogously to ([1]).

Theorem 5. The spaces $A_1(D)$ and $B_1(D)$ are isomorphic; the spaces $A_1(\overline{D})$ and $B_1(\overline{D})$ are also isomorphic.

Theorem 6. If $D\{|z_j| < R_j(z),\ j = 1, \dots, m;\ z \in H_D\}$ and $D_1\{|z_j| < R_j^{(1)}(z),\ j = 1, \dots, m;\ z \in H_{D_1}\}$ are arbitrary complete m-fold Hartogs domains with unbounded projections, having identical projections $H_D = H_{D_1}$, then the spaces $A_1(D)$ and $A_1(D_1)$ are isomorphic. The spaces $A_1(\overline{D})$ and $A_1(\overline{D}_1)$ are also isomorphic.

Theorems 5 and 6 are proved analogously to Theorems 5 and 6 of ([1]).

§ 2. Let $D\{|z_j| < R_j(z),\ j = 1, \dots, m;\ z \in H_D\}$ be an arbitrary complete m-fold Hartogs domain with planes of symmetry $z_1 = 0, \dots, z_m = 0$ over the space C^n, such that the domain $\{\log |z_j|,\ j = 1, \dots, m\}$ is convex ([2]). Consider the class $A'(D) \subset A(D)$ of functions $f(w, z)$ satisfying the condition

$$\|f(w, z)\|_r = \sup_{(w,z)\in D_r} |f(w, z)| < \infty, \qquad D_r\{(w, z) : (w/r, z) \in D\}$$

$(0 < r < 1)$. The class $A'(D)$, with the topology defined by the system of norms $\|a\|_r$, is a linear topological space. In the case when the domain D is bounded, $A'(D)$ coincides with $A(D)$. Analogously, consider the class $A'(\overline{D}) \subset A(\overline{D})$ of functions $f(w, z)$ satisfying the condition

$$\|f(w, z)\|_r = \sup_{D_r} |f(w, z)| < \infty, \qquad D_r : \{(w, z) : (w/r, z) \in D\} \quad (r > 1).$$

Let $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_j \geq 0,\ j = 1, \dots, m;\ \sum_{j=1}^{m} \alpha_j = \|\alpha\| = 1;\ |w|^\alpha =$

$$= |z_1|^{\alpha_1} \cdots |z_m|^{\alpha_m}; \qquad G_D(\alpha) = \sup_{(w,z)\in D} (|w|^\alpha), \qquad S_D\{\alpha : G_D(\alpha) < \infty\} \quad \text{(see (3))}.$$

Analogously to Lemma 1 ([3]), one can prove that the set S_D is convex.

Theorem 7. If $f(w, z) \in A'(D) \subset A(D)$, then for any k the inequality

$$|f_k(z)| \leq \|f(w, z)\|_r / r^{\|k\|} \,[G_D(k/\|k\|)]^{\|k\|} \qquad (0 < r < 1). \tag{2}$$

holds.

In particular, if $k/\|k\| \notin S_D$, then $f_k(z) \equiv 0$.

Proof. Take a number r, $0 < r < 1$, and consider the domain D_r. Let $E_\rho \subset D_r$ be an arbitrary polydisc,

$$E_\rho\{|z_j| < \rho_j,\ j = 1, \dots, m;\ z = z_0 \in H_D\}.$$

For E_ρ the Cauchy inequalities are valid:

$$|f_k(z_0)| \leq \max_{(w,z)\in \overline{E}_\rho} |f(w, z)| / \rho_1^{k_1} \cdots \rho_m^{k_m} \leq \|f(w, z)\|_r / \rho_1^{k_1} \cdots \rho_m^{k_m}.$$

Passing on the right-hand side to the lower bound over all polydiscs $E_\rho \subset D_r$, we obtain (2).

Theorem 8. If the series (1) converges in the domain D and $f(w,z) \in A'(D)$, then for every subdomain D_0, $\overline{D}_0 \subset D$, the series

$$\sum_{k/\|k\| \in S_D} \sup_{D_0} |f_k(z)| \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|}$$

converges.

Proof. Take numbers r_1 and r_2, $0 < r_1 < r_2 < 1$, so that $\overline{D}_0 \subset D_{r_1} \subset D_{r_2} \subset D$. Since $G_{D_r}(a) = rG_D(a)$, by Theorem 7 we have

$$\sum_{k/\|k\| \in S_D} \sup_{D_0} |f_k(z)| \left[G_{D_0}\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} \le \sum_{k/\|k\| \in S_D} \sup_{D_{r_1}} |f_k(z)| \left[G_{D_{r_1}}\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} \le$$

$$\le \sum_{k/\|k\| \in S_D} r_1^{\|k\|} \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} \frac{\|f(w,z)\|_{r_2}}{\left[G_{D_{r_2}}(k/\|k\|)\right]^{\|k\|}} = \|f(w,z)\|_{r_2} \sum_{k/\|k\| \in S_D} \left(\frac{r_1}{r_2}\right)^{\|k\|} < \infty.$$

Corollary. If the series (1) converges in the domain D and $f(w,z) \in A'(D)$, then in every subdomain D_0, $\overline{D}_0 \subset D$, the series

$$\sum_{k/\|k\| \in S_D} f_k(z) \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|}$$

converges uniformly.

Theorem 9. For the convergence of the series (1) in the domain D to the function $f(w,z) \in A'(D)$, it is necessary and sufficient that the series

$$\sum_{k/\|k\| \in S_D} \sup_{z \in H_D} |f_k(z)| \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} w^k \tag{3}$$

converge in the domain $D^{(1)}\{|z_j| < 1,\ j = 1, \dots, m;\ z \in H_D\}$.

Proof. Consider the domain D_r, $0 < r < 1$. Suppose that the series (3) converges in the domain $D^{(1)}$. This means that for any $r < 1$ the series

$$\sum_{k/\|k\| \in S_D} \sup_{D_r} |f_k(z)| \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} r^{\|k\|} \tag{4}$$

converges, or the series

$$\sum_{k/\|k\|\in S_D} \sup_{D_r} |f_k(z)| \left[G_{D_r}\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} \tag{5}$$

converges. Since

$$|f_k(z)w^k| \leq \sup_{D_r} |f_k(z)| \left[G_{D_r}\left(\frac{k}{\|k\|}\right)\right]^{\|k\|},$$

the series (1) converges in the domain D_r for every $r < 1$, and consequently it also converges in the domain D.

Conversely, if the series (1) converges in the domain D, then by Theorem 8, for every domain D_r $(r < 1)$ the series (5) converges, i.e., for every $r < 1$ the series (4) converges. Consequently, the series (3) converges in the domain $D^{(1)}$.

Theorem 9 is proved analogously.

Theorem 10. For the convergence of the series (1) in the domain D to the function $f(w, z) \in A'(D)$, it is necessary and sufficient that the series

$$\sum_{k/\|k\|\in S_D} f_k(z) \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} w^k$$

converge in the domain $D^{(1)}$.

Let us introduce the countably normed space $B'(D)$ of m-fold sequences of functions $a = \{f_k(z)\}$ $\left(f_k(z) \equiv 0, \text{ if } \frac{k}{\|k\|} \notin S_D\right)$ with the system of norms

$$\|a\|_r = \sum_{k/\|k\|\in S_D} \sup_{z\in H_D} |f_k(z)|\, r^{\|k\|} \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|}$$

$$(0 < r < 1).$$

We shall also consider the countably normed space $B'(\overline{D})$ of sequences of functions $a = \{f_k(z)\}$ such that, for some $r > 1$, depending in general on the sequence,

$$\|a\|_r = \sum_{k/\|k\|\in S_D} \sup_{z\in H_D} |f_k(z)|\, r^{\|k\|} \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} < \infty,$$

with the topology of the inductive limit of the Banach spaces B'_r with norms $\|a\|_r$.

Theorem 11. The spaces $A'(D)$ and $B'(D)$ are isomorphic; the spaces $A'(\overline{D})$ and $B'(\overline{D})$ are likewise isomorphic.

Proof. Theorem 10 establishes a correspondence between functions $f(w,z) \in A'(D)$ and the sequences of their Hartogs coefficients $a \in B'(D)$. This correspondence is continuous in both directions with respect to the topologies of the spaces $A'(D)$ and $B'(D)$, since for any numbers r and r_1, $0 < r < r_1 < 1$, we have

$$\|f(w,z)\|_r = \sup_{D_r} \left| \sum_{k=0}^{\infty} f_k(z) w^k \right| \le \sum_{k/\|k\| \in S_D} \sup_{D_r} |f_k(z)|\, r^{\|k\|} \left[G_D\left(\frac{k}{\|k\|}\right)\right]^{\|k\|} = \|a\|_r,$$

$$\|a\|_r \le \|f(w,z)\|_{r_1} \sum_{k/\|k\| \in S_D} \left(\frac{r}{r_1}\right)^{\|k\|}, \qquad \sum_{k/\|k\| \in S_D} \left(\frac{r}{r_1}\right)^{\|k\|} < \infty.$$

The isomorphism of the spaces $A'(\overline{D})$ and $B'(D)$ is proved analogously.

Theorem 12. If

$$D\{|z_j| < R_j(z),\ j = 1, \dots, m;\ z \in H_D\}$$

and

$$D_1\{|z_j| < R_j^{(1)}(z),\ j = 1, \dots, m;\ z \in H_{D_1}\}$$

are arbitrary complete m-circular Hartogs domains having the same projection $H_D = H_{D_1}$, then the spaces $A'(D)$ and $A'(D_1)$ are isomorphic. The spaces $A'(\overline{D})$ and $A'(\overline{D_1})$ are likewise isomorphic.

The theorem is proved by means of Theorems 10 and 11 analogously to Theorem 6 ([1]).

Corollary. If the domains D and D_1 are bounded, then the spaces $A(D)$ and $A(D_1)$ are isomorphic; the spaces $A(\overline{D})$ and $A(\overline{D_1})$ are likewise isomorphic.

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[3] M. M. Dragilev, UMN, 19, issue 1 (115) (1964).

Note: Figure translations are in progress. See original paper for figures.