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Abstract

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ON THE DIAMETER OF A STRONGLY CONNECTED GRAPH

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In this paper finite directed graphs are considered. Following ([1]), by $n(G)$, $m(G)$, $\nu(G)$ we denote respectively the number of vertices, the number of arcs, and the cyclomatic number of the graph G, and at times we shall omit the argument. Obviously, $m \leq n(n-1)$.

Recall that a directed graph G is called **strongly connected** if for every ordered pair of vertices x, y there exists a path from x to y. For a strongly connected graph G one always has $n \leq m$, and therefore $1 \leq \nu \leq (n-1)^2$.

Let $\rho(x, y)$ denote the number of arcs of a shortest path leading from x to y, and let

$$\rho(G) = \min_x \max_y \rho(x, y)$$

be the radius of the graph G, and

$$\delta(G) = \max_x \max_y \rho(x, y)$$

the diameter of this graph.

In ([2]) the inequality

$$\rho(G)\nu(G) \geq n(G) - 1. \tag{1}$$

was proved.

Hence follows the estimate* of the radius

$$\rho(G) \geq \left] \frac{n(G) - 1}{\nu(G)} \right[, \tag{1'}$$

and for every pair of integers n, ν $\left(1 \leq \nu \leq (n-1)^2\right)$ it is not difficult to give an example of a graph with $n(G) = n$, $\nu(G) = \nu$, and

$$\rho(G) = \left] \frac{n-1}{\nu} \right[.$$

Theorem. *For every strongly connected graph G for which $\nu(G) \geq 2$, the relation*

$$\delta(G)\nu(G) \geq 2(n(G) - 1) \tag{2}$$

holds.

From this inequality follows the estimate of the diameter for $\nu(G) \geq 2$:

$$\delta(G) \geq \left]\frac{2(n(G)-1)}{\nu(G)}\right[. \qquad (2')$$

Let now $\mathfrak{G}_{\nu,n}$ be the set of all strongly connected graphs G with $n(G) = n$, $\nu(G) = \nu$, and let

$$\delta(\nu, n) = \min_{G \in \mathfrak{G}_{\nu,n}} \delta(G).$$

Obviously, $\delta(1, n) = n - 1$ and $\delta((n-1)^2, n) = 1$. If $n - 1 < \nu < (n-1)^2$, then, as is easy to show, $\delta(\nu, n) = 2$.

It follows from $(2')$ that for $2 \leq \nu \leq n - 1$

$$\delta(\nu, n) \geq \left]\frac{2(n-1)}{\nu}\right[.$$

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* $]x[$ denotes the least integer a for which $x \leq a$.

Considering rosettes (see (1), p. 135) with n vertices and ν petals, we obtain

$$\delta(\nu, n) \leq \left]\frac{2(n-1)}{\nu}\right[+ 1,$$

and in the case

$$\nu \geq \frac{n - \delta - 1}{[\delta/2]},$$

where

$$\delta = \left]\frac{2(n-1)}{\nu}\right[,$$

we have

$$\delta(\nu, n) = \left]\frac{2(n-1)}{\nu}\right[.$$

Figure 1 gives an example of a rosette with $n = 8$ vertices, $\nu = 3$ petals, and diameter

Fig. 1

Figure 1: Fig. 1

$$\delta(3,8) = \left]\frac{2(8-1)}{3}\right[= 5.$$

Fig. 1

Let us note that the problem of finding an estimate for the diameter of a strongly connected graph using only the numbers of arcs and vertices of the graph, or, equivalently, the number of vertices and the cyclomatic number, was posed in (1). There also a conjecture belonging to Bratton (2) was formulated. Inequality (2′), obtained in the present paper, is stronger than Bratton' s conjecture.

Let x be some vertex of a strongly connected graph G. Construct a partial graph A that is a tree growing from the root x.* We shall always construct the tree A in such a way that condition $\mathfrak{A}$ is satisfied: the i-th level of the tree consists of all vertices y of the graph G for which $\rho(x, y) = i$. Then the height of the constructed tree is equal to

$$\rho(x) = \max_y \rho(x, y).$$

The following simple result holds.

Lemma 1. If λ is the number of terminal vertices of a tree A satisfying condition $\mathfrak{A}$, then

$$\rho(x)\lambda \geq n - 1.$$

Hence, in particular, inequality (1) follows, since always $\lambda \leq \nu(G)$.

Definition. A path μ of a graph G will be called **straight** if in the graph G from every vertex of this path, except perhaps the last, exactly one arc leaves.

Let s be the number of arcs of the longest straight path.

Lemma 2. If, for a strongly connected graph G, $\nu(G) \geq 2$ and λ is the number of terminal vertices of a covering tree A growing from the vertex x_0, the beginning of some straight path of length s, then $\nu \geq \lambda + 1$.

Proof. Since from every vertex of G at least one arc must leave, we have $\nu \geq \lambda$. Suppose $\nu = \lambda$. This means that the arcs of the graph G not belonging to the tree (their number is equal to ν) leave the terminal vertices of the tree, and moreover exactly one arc leaves each terminal vertex. Consequently, at least one of these arcs enters the root vertex x_0. But this means the existence in the graph G of a straight path of length $s + 1$, which is impossible.

We now turn to the proof of the theorem. For this purpose construct a covering tree A growing from the vertex x_0, the initial vertex of some straight path μ_0 of length s. As before, let λ denote the number of terminal vertices of the tree A. Consider two cases.

1. $s \geq \delta/2$. Then, as is easy to see,

$$n - 1 \leq s + (\delta - s)\lambda = \delta\lambda - (\lambda - 1)s,$$

and therefore

$$2(n - 1) \leq 2\delta\lambda - 2s(\lambda - 1) \leq 2\delta\lambda - \delta(\lambda - 1) = \delta(\lambda + 1).$$

* That is, in the terminology of (1), a pradtree with root x.

Using Lemma 2, we obtain

$$2(n - 1) \leq \delta\nu.$$

2. $s < \delta/2$. In the set X of all vertices of the tree A, distinguish three subsets X_1, X_2, X_3: X_1 is the set of vertices of the path μ_0; X_2 is the set of vertices x of the tree for which in A there exists a path of length $\leq s - 1$ from x to some terminal vertex of the tree (for $s = 0$ the set X_2 is empty); $X_3 = X \setminus (X_1 \cup X_2)$.

Obviously,

$$n(G) = |X| \leq |X_1| + |X_2| + |X_3|.$$

It is easy to see that

$$|X_1| = s + 1, \qquad |X_2| \leq s\lambda.$$

We shall show that $|X_3| \leq (\delta - 2s)\nu/2$. Indeed, if μ is any path in the tree A joining two vertices of the subset X_3, then, obviously, the length of this path is $\leq \delta - 2s - 1$, and therefore the number of its vertices is $\leq \delta - 2s$. Let σ denote the number of longest paths of the tree A whose initial and terminal vertices belong to X_3. It is easy to see that σ is equal to the number of vertices z in X_3 possessing the following property: every path of the tree A beginning at the vertex z contains no other vertices of the subset X_3. Denote the set of such vertices by X_3'.

Let $A(z)$ be the set of vertices of the tree A reachable in A from the vertex z. To each vertex z we associate the set $B(z)$ of arcs of the graph G which do not belong to the tree A and leave vertices of the set $A(z)$. Obviously, if $z_1 \neq z_2$, then $B(z_1) \cap B(z_2) = \emptyset$.

We shall show that $|B(z)| \geq 2$ for all $z \in X_3'$. Suppose the contrary: $|B(z_0)| = 1$ for some vertex $z_0 \in X_3'$. Then in the tree A from the vertex z_0 there issues a unique path $\mu(z_0)$, ending at a terminal vertex of the tree, and from the vertices of $\mu(z_0)$ there issues only one arc u not belonging to the tree. This arc must issue from the terminal vertex of the path $\mu(z_0)$. Let t be the vertex into which the arc u enters. Then t belongs to the set of vertices of the path $\mu(z_0)$, since otherwise this would mean the existence in the graph G of a directed path of length $s+1$. But if t is one of the vertices of the path $\mu(z_0)$, then we obtain a circuit C, from none of whose vertices there issues an arc not belonging to the circuit, which is also impossible, since the graph G is strongly connected and $\nu(G) \geq 2$. Consequently, $|B(z)| \geq 2$ for all $z \in X_3'$, and therefore $\sigma \leq \nu/2$.

Thus,

$$n \leq s + 1 + s\lambda + (\delta - 2s)\sigma \leq 1 + s(\lambda + 1) + (\delta - 2s)\nu/2,$$

$$2(n-1) \leq 2s(1+\lambda) + (\delta - 2s)\nu \leq 2s\nu + (\delta - 2s)\nu = \delta\nu.$$

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.