SOME CRITERIA FOR THE COMPLETENESS OF A SYSTEM OF ROOT VECTORS OF A LINEAR OPERATOR IN A SPACE WITH TWO NORMS

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SOME CRITERIA FOR THE COMPLETENESS OF A SYSTEM OF ROOT VECTORS OF A LINEAR OPERATOR IN A SPACE WITH TWO NORMS

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In the present note (§ 1) some theorems are established on the completeness of the system of root vectors of a completely continuous operator in a space with two norms (1), i.e., in a Banach space $\mathfrak{B}$ in which a weaker Hilbert norm is also introduced. Some conditions are also indicated for the n-fold completeness (2) of the system of eigenvectors and associated vectors of a polynomial operator pencil in such a space (§ 2). In the third section one application of the results of § 2 to elliptic operators is given. In the last section two theorems are established on the completeness of the system of generalized root vectors. The results of the note are, in the main, generalizations of some theorems on completeness in Hilbert space (2–6).

1. Let $\mathfrak{B}$ be a Banach space, $\Omega(\mathfrak{B})$ the normed ring of all linear bounded operators acting in $\mathfrak{B}$, and $\mathfrak{S}_\infty(\mathfrak{B})$ the two-sided ideal of completely continuous operators. For an operator $A \in \Omega(\mathfrak{B})$ put

$$s_{n+1}(A) = \inf \|A - K\| \qquad (n = 0, 1, \ldots),$$

where the lower bound is taken over all n-dimensional operators $K \in \Omega(\mathfrak{B})$, and

$$d_n(A) = \inf_{L_n} \sup_{\|x\| \leqslant 1} \rho(Ax, L_n) \qquad (n = 0, 1, \ldots),$$

where $\rho(z, L)$ is the distance from z to L, and the lower bound is taken over all n-dimensional subspaces $L_n \subset \mathfrak{B}$.

For $p > 0$, by $\mathfrak{S}_p(\mathfrak{B})$ ($\mathfrak{D}_p(\mathfrak{B})$) we denote the set of all operators for which $\sum s_n^p(A) < \infty$ ($\sum d_n^p(A) < \infty$). As is known, $\mathfrak{S}_p(\mathfrak{B}) \subset \mathfrak{D}_p(\mathfrak{B})$ $(p > 0)$. By $\mathfrak{N}_p$ $(0 < p \leqslant 1)$ we shall denote the set of all operators $A \in \Omega(\mathfrak{B})$ representable in the form

$$Ax = \sum_{j=1}^{\infty} \alpha_j f_j(x) y_j, \qquad (f_j \in \mathfrak{B}^*,\ y_j \in \mathfrak{B},\ \|f_j\| = \|y_j\| = 1,\ j = 0, 1, \ldots), \quad (1)$$

where $\sum |\alpha_j|^p < \infty$.

We shall call an operator $A \in \mathfrak{S}_\infty(\mathfrak{B})$ **complete** if the system of its root vectors corresponding to nonzero eigenvalues is complete in $\mathfrak{B}$. By $\mathfrak{R}(A)$ below we denote the set of values of the operator A.

Suppose that in the space $\mathfrak{B}$ a scalar product (x, y) is given, and that the norm $|x| = \sqrt{(x, x)}$ generated by it is weaker than the original norm $\|x\|$. Completing the space $\mathfrak{B}$ in the norm $|\cdot|$, we obtain a Hilbert space $\mathfrak{H}$.

An operator $A \in \Omega(\mathfrak{B})$ is called **Hermitian** if $(Ax, y) = (x, Ay)$ for all $x, y \in \mathfrak{B}$. We shall say that the operator A is $\mathfrak{H}\mathfrak{B}$-continuous (1) and write $A \in \Omega(\mathfrak{H}, \mathfrak{B})$, if there exists a number C such that,

that $\|Ax\| \leq C|x|$ $(x \in \mathfrak{B})$. An operator $A \in \Omega(\mathfrak{B})$ is called **regular** (7) if there exists an operator $A^+ \in \Omega(\mathfrak{B})$ such that $(Ax, y) = (x, A^+ y)$ $(x, y \in \mathfrak{B})$. The set of all regular operators will be denoted by Π.

Theorem 1. *Let A be an $\mathfrak{H}\mathfrak{B}$-continuous Hermitian operator and let $A \in \mathfrak{D}_p(\mathfrak{B})$ for some $p > 0$. If T is a regular operator from $\mathfrak{S}_\infty(\mathfrak{B})$, $B = (I + T)A$, and $\overline{\mathfrak{R}}(B) = \mathfrak{B}$, then the operator B is complete in $\mathfrak{B}$.*

This theorem is proved on the basis of a theorem of M. V. Keldysh (2) (see also (3)) and the following simple propositions.

1°. *If $A \in \Omega(\mathfrak{H}, \mathfrak{B})$, $\overline{\mathfrak{R}}(A) = \mathfrak{B}$, and the operator A is complete in $\mathfrak{H}$, then A is complete also in $\mathfrak{B}$.*

2°. *If the operator A is Hermitian and $A \in \mathfrak{D}_p(\mathfrak{B})$, then $A \in \mathfrak{S}_r(\mathfrak{H})$ for every $r > p$.*

Remark 1. The condition $A \in \mathfrak{D}_p(\mathfrak{B})$ in Theorem 1 may be replaced by one of the following: a) $A^n \in \mathfrak{R}_1(\mathfrak{B})$ for some natural number n; b)

$$\sum_{k=1}^{\infty} |\lambda_k|^p < \infty,$$

where $\{\lambda_k\}$ is the complete system of eigenvalues of the operator A.

We shall next need the following proposition on estimating the resolvent of an $\mathfrak{H}\mathfrak{B}$-continuous operator in the space $\mathfrak{H}$.

Lemma 1. *If A is an $\mathfrak{H}\mathfrak{B}$-continuous operator and $A \in \mathfrak{D}_p(\mathfrak{B})$ (respectively $A \in \mathfrak{R}_q(\mathfrak{B})$), then*

$$\lim_{\rho \to \infty} \left\{ \rho^{-r} \ln \max_{|\lambda| = \rho} \left| (I - \lambda A)^{-1} \right| \right\} = 0$$

for every $r > p$ (respectively for every $r > (1/q - 1/2)^{-1}$).

With the aid of Lemma 1 and results of A. S. Markus (^{5}) (lemma and Theorem 8), the following is established.

Theorem 2. *Let the operator $A \in \Omega(\mathfrak{H}, \mathfrak{B})$, $|\arg(Ax, x)| \leq \gamma\pi/2$ $(\gamma < 2)$ for all $x \in \mathfrak{B}$, and let at least one of the conditions be fulfilled: a) $A \in \mathfrak{D}_p(\mathfrak{B})$ $(p < \gamma^{-1})$; b) $A \in \mathfrak{R}_q(\mathfrak{B})$ $(q < (\gamma + 1/2)^{-1})$. If T is a regular operator from $\mathfrak{S}_\infty(\mathfrak{B})$, $B = (I + T)A$, and $\overline{\mathfrak{R}}(B) = \mathfrak{B}$, then the operator B is complete in $\mathfrak{B}$.*

2. An operator $A \in \Pi$ will be called **normal** if $A^+A = AA^+$.

Theorem 3. *Let $A \in \Omega(\mathfrak{H}, \mathfrak{B})$ be a normal operator, and suppose that A^n is Hermitian for some natural number n. If $A \in \mathfrak{D}_p(\mathfrak{B})$ $(p > 0)$ and $\overline{\mathfrak{R}}(A) = \mathfrak{B}$, then the system of eigenvectors and associated vectors of each of the pencils*

$$L(\lambda) = I - \sum_{j=0}^{n-1} \lambda^j A_j^{jT} - \lambda^n A^n, \qquad M(\lambda) = I - \sum_{j=0}^{n-1} \lambda^j T_{jA}^{j} - \lambda^n A^n,$$

where T_j $(j = 0, 1, \dots, n-1)$ are regular operators from $\mathfrak{S}_p(\mathfrak{B})$, is n-fold complete in $\mathfrak{B}$.

The proof of this theorem is based on propositions 1°, 2° and the theorem of M. V. Keldysh (^{2}).

Remark 2. With the aid of a theorem of Yu. A. Palant (^{4}) it can be shown that Theorem 3 remains valid for the pencil $L(\lambda)$ when the condition $A \in \mathfrak{D}_p(\mathfrak{B})$ is replaced by the following conditions: $A \in \mathfrak{S}_\infty(\mathfrak{B})$; $T_0 A^n$, $A_k^{kT} \in \mathfrak{D}_p(\mathfrak{B})$ $(k = 1, \dots, n-1)$.

With the aid of Lemma 1 and the results of the article (^{6}), the following is proved.

Theorem 4. *Let the operator $A \in \Omega(\mathfrak{H}, \mathfrak{B})$, $|\arg(Ax, x)| \leq \gamma\pi/2$ $(\gamma < 2)$ for all $x \in \mathfrak{B}$, and let at least one of the conditions be fulfilled: a) $A \in \mathfrak{D}_p(\mathfrak{B})$ $(p < \gamma^{-1})$; b) $A \in \mathfrak{R}_q(\mathfrak{B})$ $(q < (\gamma + 1/2)^{-1})$. If T_j $(j = 0, 1, \dots, n-1)$ are regular operators from $\mathfrak{S}_\infty(\mathfrak{B})$ and $\overline{\mathfrak{R}}(A) = \mathfrak{B}$, then the system of eigenvectors and associated vectors of each of the pencils*

$$I - T_0 - \lambda A T_1 - \cdots - \lambda^{n-1} A T_{n-1} - \lambda^{nA}, \qquad I - T_0 - \lambda T_1 A - \cdots - \lambda^{n-1} T_{n-1} A - \lambda^{nA}$$

is n-fold complete in $\mathfrak{B}$.

3. Let D be a bounded domain with a sufficiently smooth boundary Γ in m-dimensional Euclidean space, and

$$M_\lambda(u) = \sum_{i,k=1}^{m} p_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} q_i \frac{\partial u}{\partial x_i} + (r_0 + \lambda r_1 + \dots + \lambda^n r_n) u,$$

where p_{ik} $(i,k = 1,\dots,m)$, q_i $(i = 1,\dots,m)$, and r_j $(j = 1,\dots,n)$ are twice continuously differentiable functions in D, p_{ik} and r_n are real, and $r_n(x) > \alpha$,

$$\sum_{i,k=1}^{m} p_{ik}\xi_i\xi_k > \beta \sum_{i=1}^{m} \xi_i^2 \quad (x \in D,\ \alpha > 0,\ \beta > 0).$$

On the basis of Theorem 3 one obtains

Theorem 5. *The system of eigenfunctions and associated functions of the operator pencil* $M_\lambda(u)$*, under the boundary condition* $u|_\Gamma = 0$*, is* n*-fold complete in the space* $\overset{\circ}{W}{}_2^2(D)$*.*

We note that the n-fold completeness of the system of eigenfunctions and associated functions of the indicated pencil in the space $L_2(D)$ was established by M. V. Keldysh (^{2}).

4. In this section we shall show that, if the condition of $\mathfrak{H}\mathfrak{B}$-continuity of the operators under consideration is abandoned, then in some cases one can establish completeness of the system of generalized root vectors.

A functional $f \in \mathfrak{B}^*$ will be called a **generalized root vector** of the operator $A \in \Omega(\mathfrak{B})$, corresponding to the eigenvalue λ, if $f((A-\lambda I)^n x) = 0$ for all $x \in \mathfrak{B}$ for some natural number n. We shall say that the system of generalized root vectors of the operator A is complete if $f(x_0) = 0$ for all generalized root vectors f of the operator A only when $x_0 = 0$ (cf. (^{8}), Ch. I, § 4).

Theorem 6. *If a Hermitian operator* $A \in \mathfrak{S}_\infty(\mathfrak{B})$*, then the system of generalized root vectors of the operator* A *is complete.*

This theorem is proved on the basis of results of M. G. Krein (^{1}). We note that every generalized root vector of a Hermitian operator A, corresponding to a nonzero eigenvalue, is a generalized eigenvector.

Lemma 2. *If, for the operator* $A \in \mathfrak{S}_\infty(\mathfrak{B})$ $(A \neq 0)$*, the following conditions are fulfilled:*

1) $|\arg(Ax,x)| \leq \gamma\pi/2$ $(\gamma < 2)$ for all $x \in \mathfrak{B}$;

2)
$$\lim_{\rho\to\infty} \left\{ \rho^{-l} \ln \max_{|\lambda|=\rho} \|(I-\lambda A)^{-1}x\| \right\} = 0 \quad (l < \gamma^{-1})$$

for all $x \in \mathfrak{B}$*, then* A *has at least one nonzero eigenvalue.*

With the aid of Lemma 2 the following theorem is proved (cf. (^{3}), p. 107).

Theorem 7. *If the operator* $A \in \mathfrak{S}_\infty(\mathfrak{B})$, $|\arg(Ax,x)| \leq \gamma\pi/2$ $(\gamma < 2)$ *for all* $x \in \mathfrak{B}$*, and at least one of the conditions is fulfilled: a)* $A \in \mathfrak{S}_p(\mathfrak{B})$ $(p < \gamma^{-1})$*; b)* $A \in \mathfrak{R}_q(\mathfrak{B})$ $(q = (\gamma + \frac{1}{2})^{-1})$*, then the system of generalized root vectors of the operator* A *is complete.*

We note that condition b) of Theorem 7 may be replaced by the following: the operator A admits the representation (1), where $\alpha_j = o(j^{-1/q})$.

Remark 3. Under the conditions of Theorem 6 (Theorem 7), one can also assert that if $f(x_0) = 0$ for all generalized eigen (root) vectors f of the operator A corresponding to nonzero eigenvalues, then $Ax_0 = 0$.

Remark 4. If the space $\mathfrak{B}$ is reflexive, then under the conditions of Theorems 6 and 7 the system of root vectors of the operator A^* is complete in $\mathfrak{B}^*$, and the system of its root (in Theorem 6—eigen) vectors corresponding to non-

corresponding to zero eigenvalues, is complete in $\mathfrak{R}(A^*)$. We note that, at the same time, the system of root vectors of the operator A itself will, generally speaking, not be complete even in $\mathfrak{R}(A)$ (cf. the example in (9)).

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Note: Figure translations are in progress. See original paper for figures.

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