SET-THEORETIC OPERATIONS

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Abstract

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SET-THEORETIC OPERATIONS AND RECURSIVE HIERARCHIES

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A general definition is given of a recursive hierarchy (with ω_1* stages) in an arbitrary space. This definition may be regarded as an analogue of the definition of the classical hierarchies (^{6}), which makes it possible to apply methods of the theory of operations on sets to establish a number of properties of recursive hierarchies. The hyperarithmetical (^{9}) and ω_1-stage $\mathfrak{S}$-hierarchies (^{8}) turn out to be equivalent in the sense specified in Theorems 5 and 6 to the effective (eff.) B- and, respectively, eff. C-hierarchies**, obtained in certain particular cases from our general definition***.

1. K-numberings. $N = \{0, 1, 2, ...\}$; e, i, m, n are variables on N. Subsets of N are called **chains**, and sets of chains are called **bases**. A collection $\mathfrak{M}$ of arbitrary objects, supplied with some numbering α, is called a **family** and is denoted $\langle \mathfrak{M}, \alpha \rangle$, or simply α. Let $\langle \mathfrak{M}_1 \alpha_1 \rangle$ and $\langle \mathfrak{M}_2 \alpha_2 \rangle$ be two arbitrary families; let $P(x, y)$ be a predicate defined on $\mathfrak{M}_1 \times \mathfrak{M}_2$. We shall say: "for every x ($\in \mathfrak{M}_1$) one can find y ($\in \mathfrak{M}_2$) such that $P(x, y)$,"meaning by this: "there exists a partial recursive function (p.r.f.) $f(n)$ such that if n is an α_1-number of some $x \in \mathfrak{M}_1$, then $f(n)$ is defined and is equal to the α_2-number of such a $y \in \mathfrak{M}_2$ that $P(x, y)$." The family of n-ary p.r.f.' s is assumed to be supplied with a Gödel numbering (^{1}); for $n = 1$, the p.r.f. with number e is denoted by $\langle e \rangle$; o is the one-place nowhere-defined p.r.f. The family of all recursively enumerable subsets of N is supplied with the numbering τ: $\forall e . \tau e =$ the domain of definition of the p.r.f. $\langle e \rangle$. For arbitrary families α and β, $\alpha \leq [e]\beta$ (and $\alpha \simeq [e]\beta$) means that the numbering α is reduced to the numbering β by means of the p.r.f. (recursive permutation) $\langle e \rangle$, i.e. $\alpha n = \beta \langle e \rangle n$; $\alpha \leq \beta \iff (\exists e)(\alpha \leq [e]\beta)$; and $\alpha \simeq \beta \iff (\exists e)(\alpha \simeq [e]\beta)$. If $\{\alpha_n\}_{n \in A}$ and $\{\beta_m\}_{m \in B}$ $(A, B \subseteq N)$ are two sequences of families and for any $n \in A$, $m \in B$ one can find a number $f(n, m)$ such that $\alpha_n \leq [f(n, m)]\beta_m$, then we shall say that $\alpha_n \leq \beta_m$ uniformly in n, m. A simple numbering α will be called a K-numbering if
$(\forall e_1, e_2)(\langle e_1 \rangle = \langle e_2 \rangle \Rightarrow \alpha e_1 = \alpha e_2)$. If m_0 is some number of the p.r.f. o, then αm_0 will be denoted by o_α. For example, the family τ is K-numbered. The

numbering $\langle e\rangle$ is effectively complete (^{5}). Denote by e^* such a general recursive function (g.r.f.) that for every e, $\langle e^*\rangle$ is also a g.r.f. and for every n

$$\langle\langle e^*\rangle(n)\rangle = \begin{cases} \langle\langle e\rangle(n)\rangle, & \text{if } \langle e\rangle(n) \text{ is defined,} \\ o, & \text{otherwise;} \end{cases}$$

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[*] ω_1 is the least nonconstructive ordinal number (^{10}).

[**] This clarifies the connection between $\mathfrak{S}$- and eff. C-hierarchies (^{8}).

[***] For the concepts and notation used below see (^{1},^{4}) or (^{5},^{6}).

then if α is a K-enumeration, then

$$\alpha\langle e^*\rangle(n) = \begin{cases} \alpha\langle e\rangle(n), & \text{if } \langle e\rangle(n) \text{ is defined,} \\ o_\alpha & \text{otherwise.} \end{cases} \tag{1}$$

2. Standard operators. Let E be an arbitrary set, called a space; let Ψ_M be a set-theoretic operation (s.t.o.) with base M. A function which assigns to each simply enumerated family α of subsets of the space E a family $\beta = H_M\alpha$, defined as follows: $\forall e.\, \beta e = \Psi_M\{\alpha\langle e^*\rangle(n)\}$, will be called a **standard operator with base** M. From (1) it is clear that if α is a K-enumerated family, then $H_M\alpha$ is the family of all sets obtained as a result of the s.t.o. Ψ_M applied to all possible enumerated sequences of sets of the family α. A **composite standard operator with base** is a function which assigns to a sequence $\{\alpha_n\}_{n\in N}$ of simply enumerated families the family $\beta = H_M\{\alpha_n\}$, where $\forall e.\, \beta e = \Psi_M\{\alpha_n\langle e^*\rangle(n)\}$. If all

$$\alpha_n = \alpha, \quad \text{then } H_M\{\alpha_n\} = H_M\alpha.$$

In what follows, all K-enumerated families are families of subsets of the space E.

Theorem 1. *If all α_n, $n \in N$, are K-enumerated families, then $H_M\{\alpha_n\}$ is also a K-enumerated family.*

Theorem 2. *If for each n the families β_n are simple, and α_n are K-enumerated, then for any general recursive function $\langle e\rangle$ such that $\alpha_n \leqslant [\langle e\rangle(n)]\beta_n$, one can find a number $h(e)$ such that*

$$H_M\{\beta_n\} \leqslant [h(e)]H_M\{\alpha_n\}.$$

Let a general recursive function $\sigma(m,n)$ map N^2 one-to-one onto N, and let $(M \mid L_i)$ be the base of such an s.t.o. that, for any sequence $\{a_i\}_{i\in N}$ of families of sets,

$$\Psi_M\{\Psi_{L_i}\{a_{in}\}\} = \Psi_{M|L}\{\beta m\},$$

where $\beta\sigma(i,n) = a_{in}$.

Theorem 3. *There exists a number* e_0 *such that, for any* K*-enumerated family* α *and any bases* $M, L_0, \dots, L_i, \dots$*,*

$$H_M\{H_{L_i}\alpha\} \cong [e_0]H_{(M|L_i)}\alpha.$$

3. Definition of recursive hierarchies. Let $(O, <_0)$ be Kleene' s system of notations for constructive ordinal numbers (^{10}). Let $a' = 2^a$, if $a > 0$, and let the partial recursive function $[a]_i$ be such that if $a \in O$ and $a = 2^{(a)_0}$, then for every i, $[a]_i$ is defined,

$$[a]_i <_0 [a]_{i+1} <_0 a \quad \text{and} \quad \lim_i |[a]_i| = |a|.$$

Let $\mathscr{H}r(n)$ be such a general recursive predicate that

$$(n \in O \text{ and } \mathscr{H}r(n)) \iff (|n| \text{ is a limit nonzero ordinal number}).$$

We shall say that a base M **has property** (A) if $N \in M$ and all chains belonging to M are infinite. By M^c is denoted the base of the s.t.o. complementary to Ψ_M.

Basic definition. Let the base M or M^c have property (A), and let λ be a K-enumerated family of subsets of the space E, with $\emptyset$ and E belonging to λ. The sequence $\{\lambda_a\}_{a\in O}$ of families of subsets of the space E, defined as follows:

1) $\lambda_1 = \lambda$;

2)

$$\lambda_{a'} = \begin{cases} H_M\lambda_a, & \text{if } \mathscr{H}r(a'), \\ H_{M^c}\lambda_a & \text{otherwise;} \end{cases}$$

3) if $a \neq 2^{(a)_0}$, then

$$\lambda_a = H_{M^c}\{\lambda_{[a]_i}\}$$

is called a **recursive hierarchy** (r.h.) with base M and initial family λ, or, more briefly, an $\langle M, \lambda\rangle$-hierarchy.

It follows from Theorem 1 that each λ_a, $a \in O$, is a K-numbered family. By $\bar{\alpha}$ we shall denote the family of complements of the sets of the family α: $\bar{\alpha}n = E \setminus \alpha n$. Theorems 4–6 are proved by effective induction over $(O, <_0)$ (^{11}).

Theorem 4. For any $a, b \in O$, $a <_0 b \Rightarrow \lambda_a \leq \lambda_b$ uniformly in ab; if, moreover, $\bar{\lambda} \leqslant' H_M\lambda$, then $a <_0 b \Rightarrow \lambda_a \leqslant \lambda_b$ uniformly in a, b.

4. Relation with the hyperarithmetical and $\mathfrak{S}$-hierarchies. Here $E = N$. Let, for $a \in O$, the predicates $H_a(x)$ and $\mathfrak{S}_a(x)$ be defined as in (8,9); put

$$h_a n = x(\exists y)T_1^{H_a}(n, x, y), \qquad \mathfrak{h}_a n = x(\forall f)(\exists y)T_1^{\mathfrak{S}_{a'} f}(n, x, y).$$

Theorem 5. Let M be a base of the t.m.o. of intersection, and let $\{\tau_a\}_{a\in O}$ be an r.u. with base M and initial family τ. Then for every $a \in O$

$$\tau_a \simeq \begin{cases} h_{a'}, & \text{if } \mathfrak{H}r(a'), \\ \bar{h}_{a'}, & \text{otherwise} \end{cases}$$

uniformly in a.

Theorem 6. Let M be a complete base of the A-operation and let $\lambda = H_M\mathfrak{ct}$. Then the $\langle M, \lambda\rangle$-hierarchy is such that, for $a \in O$,

$$\lambda_a \simeq \begin{cases} \mathfrak{h}_{a'}, & \text{if } \mathfrak{H}r(a'), \\ \bar{\mathfrak{h}}_{a'}, & \text{otherwise} \end{cases}$$

uniformly in a.

5. Properties of recursive hierarchies. We denote the totality of all chains by Y and shall regard it as a T_0-space with the following topology (7): for any finite chain ξ', the set $\sigma_{\xi'}$ of all chains containing ξ' as a subset is called **elementarily open**; an **open set** is any union of elementarily open sets. Let $\alpha = \{\alpha n\}$, $\alpha n \subseteq E$, $n \in N$; the function $F : E \to Y$, defined as follows:

$$\forall x \in E \ . \ F(x) = n \quad (x \in \alpha n),$$

will be called the **component function** corresponding to the sequence α. From the definition of a t.m.o. it is clear that $\Psi_M\{\alpha n\} = F^{-1}(M)$, where F is the component function corresponding to α. The class of continuous mappings of Y into itself coincides with the class of component functions corresponding to all possible sequences of open sets. An **effectively open set** will mean any enumerable union of elementarily open sets; denote by γ the numbering of the family of effectively open sets induced by the numbering τ. A component function $F : Y \to Y$ corresponding to some γ-enumerable sequence of effectively open sets will be called a **computable operation**. This definition is equivalent to the definition of a computable operation (7).

Theorem 7 (fixed point theorem). If the function $F : Y \to Y$ is continuous, then there exists a chain ξ_0 such that $F(\xi_0) = \xi_0$. If, moreover, F is a computable operation, then ξ_0 is a recursively enumerable subset of N.

Let $\langle \mathfrak{M}, \alpha\rangle$ be a family of bases; a base $L \in \mathfrak{M}$ will be called **universal** in the family $\langle \mathfrak{M}, \alpha\rangle$ if, for any base $M \in \mathfrak{M}$, one can find a computable operation F such that $M = F^{-1}(L)$.

Theorem 8. Let $\{\lambda_a\}_{a\in O}$ be an r.u. with base M and initial family λ in the space E, and let $\{\gamma_a\}_{a\in O}$ be an r.u. with the same base M and initial family γ of effectively open sets in Y. Then for every $a \in O$, in the family γ_a one can find a base L_a such that: 1) L_a is universal in γ_a; 2) if $\gamma_a \leq H_M\gamma$, then for every $b <_0 a$, L_a does not belong to γ_b; 3) $\lambda_a \simeq H_{L_a}\lambda$ uniformly in a.

Remark. 2) follows from (1) and Theorem 9; thus, the "non-emptiness" of the families of $\langle M, \gamma\rangle$-hierarchies is established by means of the fixed point theorem instead of the traditional diagonal method.

A simply numbered family α is called **e.f.f. closed with respect to enumerable sums** if for every e one can find a number $s(e)$ such that

$$\bigcup_{n\in\tau e} \alpha n = \alpha s(e).$$

Analogously one defines **e.f.f. closedness with respect to pairwise intersections**.

Theorem 9 (an analogue of Kolmogorov' s "empty classes" theorem (2)). *Let $\langle M, \lambda\rangle$-hierarchy be such that λ is an e.f.f. closed with respect to enumerable sums and pairwise intersections, and in λ there exists an enumerable sequence of nonempty, pairwise disjoint sets. Then, for any $a \in O$, in the family λ_a one can find a set that belongs to no λ_b with $b <_O a$.*

The hypotheses of Theorem 9 are satisfied, for example, when $E = N$ and $\lambda = \tau$, or E is Baire space and λ is a family of e.f.f. open sets (3).

We shall say that a sequence $\{\alpha_a\}_{a\in O}$ of families is **cofinal** in a sequence $\{\beta_b\}_{b\in O}$ if, for every $a \in O$, one can find a number $h(a)$ such that $\alpha_a \leqslant \beta_{h(a)}$ uniformly in a.

Theorem 10. *Consider two r.s. in a space E, having bases L and M, and one and the same initial family λ. If the base L belongs to one of the families of the $\langle M, \gamma\rangle$-hierarchy in the space Y, then the $\langle L, \lambda\rangle$-hierarchy is cofinal in the $\langle M, \lambda\rangle$-hierarchy.*

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