ON THE POLES AND THE RADIUS OF CIRCULAR UNIVALENCE OF THE MEROMORPHIC CLASS \(\Phi\)

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MATHEMATICS

PAVEL TODOROV

ON THE POLES AND THE RADIUS OF CIRCULAR UNIVALENCE OF THE MEROMORPHIC CLASS Φ

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In ([1]) we considered one of the subclasses of the meromorphic class Φ:

$$f(z)=\sum_{k=1}^{n}\sum_{s=1}^{m_k}\frac{A_{ks}}{(z-a_k)^s}, \tag{1}$$

$$a_k=\delta_k e^{i\varphi_k},\quad \delta_k>0,\quad m_k\geq 1\text{—integers},$$

$$\arg A_{ks}-(s+1)(\pi+\arg a_k)\equiv \text{const}\equiv\varphi \pmod{2\pi} \tag{2}$$

for all k, s with $A_{ks}\neq 0$, and established the radius of a circle with center at $z=0$ such that every function of the subclass under consideration is univalent in it. In the present study we solve the problem completely.

Theorem 1. *Let $R(\delta_k, m_k)$ be the greatest of the numbers R such that every function of the class Φ is univalent in the circle $|z|\leq R$. Then*

$$R(\delta_k,m_k)=\min_k \delta_k\sin\frac{\pi}{2(m_k+1)}. \tag{3}$$

The proof of Theorem 1 is based on two lemmas and two theorems.

Lemma 1 *(a generalization of the lemma from ([1])). Let $p_1, p_2, \dots, p_n$, $n\geq 1$, be arbitrary real numbers, and suppose the powers $(1-z_k)^{p_k}$, $k=1,2,\dots,n$, have their principal value. Then the real part of the function*

$$P(z_1,\dots,z_n)=(1-z_1)^{p_1}\cdots(1-z_n)^{p_n}$$

is positive for arbitrary complex numbers $z_1,\dots,z_n$ lying in the circle

$$|z| < \sin\frac{\pi}{2p}, \quad p = \sum_{k=1}^{n} |p_k|$$

when $p > 1$, and in the circle $|z| < 1$ when $0 < p \leq 1$. The numbers $\sin(\pi/2p)$ *for $p > 1$ and 1 for $p \leq 1$ cannot be replaced by larger ones.*

Proof. From the inequalities $|\arg(1-z)| < \gamma$ for $|z| < \sin\gamma$, $0 < \gamma_0 \leq \pi/2$, it follows that

$$|\arg P(z_1, \dots, z_n)| < \gamma \sum_{k=1}^{n} |p_k| = p\gamma.$$

Putting here $\gamma = \gamma_0 = \pi/2p$ when $p > 1$, or $\gamma = 1$ when $0 < p \leq 1$, we obtain the first assertion of the lemma. To prove the unimprovability of these bounds for $p > 1$, set, for $p_k > 0$, $z_k = \sin(\pi/2p)e^{-i\alpha}$, where $\alpha = (\pi/2p)(p-1)$, and, for $p_k < 0$, $z_k = \sin(\pi/2p)e^{i\alpha}$; it is easy to verify that in this case $\arg P(z_1, \dots, z_n) = \pi/2$. If $p \leq 1$ and at least one of the numbers z_k is equal to 1, then either $P(z_1, \dots, z_n) = 0$, or P has no meaning. Thus both circles in the lemma are the largest possible with the required properties.

Lemma 2. *Let the function $f(z)$ be holomorphic in the circle $|z - z_0| < r_0$, univalent in the closed circle $|z - z_0| \leq c$ $(< r_0)$, and let on its circumference $|z - z_0| = c$ the derivative $f'(z) \neq 0$. Then there exists at least one $c' > c$ such that $f(z)$ is univalent in the closed circle $|z - z_0| \leq c$.*

Proof. Otherwise, for any sequence $\varepsilon_n \to 0$ as $n \to \infty$ $(\varepsilon_n > 0)$, there will be found a sequence of pairs of points $z_1^{(n)}, z_2^{(n)}$ such that $z_1^{(n)} \neq z_2^{(n)}$, $|z_j^{(n)} - z_0| \leq c + \varepsilon_n$ $(j = 1, 2)$, and $f(z_2^{(n)}) = f(z_1^{(n)})$.

Let $z_j^{(n_k)}$ be a convergent subsequence, $z_j^{(n_k)} \to z_j^*$. Clearly z_j^* lies in the disk $|z - z_0| \leq c$, and by the continuity of $f(z)$ we have $f(z_1^*) = f(z_2^*)$. Since f is univalent in the disk $|z - z_0| \leq c$, it follows that $z_1^* = z_2^*$. Consequently, in every neighborhood of z_1^* there lies a pair of points $z_1^{(n_k)}$ and $z_2^{(n_k)}$ such that $z_1^{(n_k)} \neq z_2^{(n_k)}$ and $f(z_1^{(n_k)}) = f(z_2^{(n_k)})$. Under the assumptions made about the univalence of f, the point z_1^* cannot lie in the disk $|z - z_0| < c$. However, the case $|z - z_0| = c$ is impossible, since $f'(z_1^*) \neq 0$, and therefore $f(z)$ is univalent in a neighborhood of z_1^*. The lemma is proved.

Theorem 2. *The disk $|z| \leq R(\delta_k, m_k)$ (where $R(\delta_k, m_k)$ is defined by equality (3)) is a domain of univalence for every function of the class Φ.*

Proof. For $z_1 \neq z_2$ the equality holds

$$\frac{-e^{-i\varphi}(f(z_2) - f(z_1))}{z_2 - z_1} = \sum_{k=1}^{n} \sum_{s=1}^{m_k} \frac{s}{\delta_k^{s+1}} |A_{ks}| \int_0^1 \frac{dt}{(1 - z/a_k)^{s+1}}, \tag{4}$$

where $z = (1-t)z_1 + tz_2$, and it is assumed that the segment $z_1 z_2$ does not pass through a pole of $f(z)$. According to Lemma 1, the real parts of the integrand functions are positive when

$$\left|\frac{z}{a_k}\right| < \sin\frac{\pi}{2(m_k+1)} \le \sin\frac{\pi}{2(s+1)}, \qquad k = 1, \dots, n;\ s = 1, \dots, m_k. \quad (5)$$

Consequently, for $|z_1|, |z_2| < \min_k \delta_k \sin \pi/2(m_k+1)$ (recall that $\delta_k = |a_k|$) the right-hand side of equality (4) is nonzero. Thus, for $z_1 \ne z_2$ we have $f(z_1) \ne f(z_2)$, as was required to prove.

By the criterion given in Lemma 2, the disk $|z| \le R$ will be the largest disk in which $f(z)$ is univalent if on its circumference there is at least one point z_0 such that $f'(z_0) = 0$. In the following theorem all functions of the class Φ with this property are described.

Theorem 3. *Let an arbitrary function of the class Φ have a zero on the circumference of the disk of radius R with center at $z = 0$. Then $f(z)$ has the form*

$$e^{-i\varphi} f(z) = \quad (6)$$

$$= \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_\sigma} \frac{d_{k\sigma} \exp\left\{ i(m_\sigma+1)\left(\pi + \varphi_{11} + \frac{\pi\varepsilon_{11} m_1}{2(m_1+1)} - \frac{\pi\varepsilon_{k\sigma} m_\sigma}{2(m_\sigma+1)}\right)\right\}}{\left\{ z - R \operatorname{cosec} \frac{\pi}{2(m_\sigma+1)} \exp\left(i\varphi_{11} + \frac{i\pi\varepsilon_{11} m_1}{2(m_1+1)} - \frac{i\pi\varepsilon_{k\sigma} m_\sigma}{2(m_\sigma+1)}\right)\right\}^{m_\sigma}},$$

where

$$\sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_\sigma} \varepsilon_{k\sigma} m_\sigma d_{k\sigma} \left\{ \frac{1}{R} \lg \frac{\pi}{2(m_\sigma+1)} \right\}^{m_\sigma+1} = 0. \quad (7)$$

Only for these functions is the closed disk $|z| \le R$ the largest disk of univalence.

Proof. For the derivative of a function of the class Φ we have

$$-e^{-i\varphi} f'(z) = \sum_{k=1}^{n} \sum_{s=1}^{m_k} \frac{s}{\delta_k^{s+1}} |A_{ks}| \left(1 - \frac{z}{a_k}\right)^{-s-1}. \quad (8)$$

I. First of all, if for some z with $|z| = R$ we have $f'(z) = 0$, then $A_{ks} = 0$ for $s < m_k$, $k = 1, \dots, n$. Indeed, from $|z| \le |a_k| \sin\frac{\pi}{2(m_k+1)} <$

$< \delta_k \sin\dfrac{\pi}{2(s+1)}$ and $|z| \le R$ it follows that
$\operatorname{Re}\{(1 - z_n/a_k)^{-s-1}\} > 0$, i.e. $f'(z) \ne 0$ for $|z| = R$, and $A_{ks} \ne 0$ with $s < m_k$.

II. Further, if $f'(z)$ vanishes on the circle $|z| = R$, then

$$\delta_k \sin \frac{\pi}{2(m_k+1)} = R \tag{9}$$

for all k (otherwise, among the nonnegative quantities $\mathrm{Re}(1 - z/a_k)^{-s-1}$ there would be a strictly positive one, and the equality $f'(z) = 0$ would be impossible for $|z| = R$).

III. From I and II we find the form of those functions of the class Φ whose first derivative vanishes on the circle $|z| = R$:

$$f(z) = \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_\sigma} \frac{A_{k\sigma}}{(z - a_{k\sigma})^{m_\sigma}}, \tag{10}$$

where $a_{k\sigma} \neq a_{\sigma k}$ for $k \neq \sigma$, $a_{k\sigma} = \delta_\sigma e^{i\varphi_{k\sigma}}$ for $k = 1, \dots, \nu_\sigma$; $\delta_\sigma < \delta_a$ and $m_\sigma < m_a$ for $\sigma < a$. The derivative of a function of the form (10) is equal to (we put $A_{k\sigma} = d_{k\sigma} \exp(i\beta_{k\sigma})$)

$$-e^{-i\varphi} f'(z) = \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_\sigma} m_\sigma d_{k\sigma} \left\{ \delta_\sigma \left(1 - \frac{z}{a_k}\right) \right\}^{-m_\sigma - 1}. \tag{11}$$

According to Lemma 1, from the equality $f'(\zeta) = 0$, $\zeta = Re^{i\varphi}$, it follows that

$$\zeta = \delta_\sigma e^{i\varphi_{k\sigma}} \sin \frac{\pi}{2(m_\sigma+1)} \exp\left(\frac{i\pi m_\sigma \varepsilon_{k\sigma}}{2(m_\sigma+1)}\right), \quad k = 1, \dots, \nu_\sigma;\ \sigma = 1, \dots, \tau, \tag{12}$$

where $\varepsilon_{k\sigma} = \pm 1$ must not have a constant sign. From (9), (11), and (12) we obtain (7). Further, from (12) we have

$$\varphi_{k\sigma} \equiv \varphi_{11} + \frac{\pi \varepsilon_{11} m_1}{2(m_2+1)} - \frac{\pi \varepsilon_{k\sigma} m_\sigma}{2(m_\sigma+1)} \pmod{2\pi}. \tag{13}$$

Further,

$$\zeta = R \exp\left(i\varphi_{11} + \frac{i\varepsilon_{11}\pi m_1}{2(m_1+1)}\right) = \frac{1}{2} a_{11} \left(1 - \exp\left(-\frac{i\pi\varepsilon_{11}}{m_1+1}\right)\right).$$

From the condition that a function of the form (10) belongs to the class Φ, and from (13), we find

$$\beta_{k\sigma} \equiv \varphi + (m_\sigma + 1)\left(\pi + \varphi_{11} + \frac{\pi\varepsilon_{11} m_1}{2(m_1+1)} - \frac{\pi \varepsilon_{k\sigma} m_\sigma}{2(m_\sigma+1)}\right) \pmod{2\pi}. \tag{14}$$

Substituting (9), (13), and (14) into (10), we obtain (6), as was required to prove.

If we put $m_1 = \max_\sigma m_\sigma$, then from (6) it immediately follows that $\tau = 1$, $k = 1, \dots, \nu_1$, and (5) becomes $\sum_{k=1}^{\nu_1} \varepsilon_{k1} d_{k1} = 0$. Hence $\sum_k' d_{k1} = \sum_k'' d_{k1} = d$, where $\sum_k'$, $\sum_k''$ denote sums over those k for which $\varepsilon_{k1} = +1$, respectively $\varepsilon_{k1} = -1$. Consequently, (6) turns into

$$e^{-i\varphi} f(z) = \tag{6'}$$

$$= \frac{d \exp(i(m_1+1)(\pi+\varphi_{11}))}{\left(z - R \operatorname{cosec} \frac{\pi}{2(m_1+1)} e^{i\varphi_{11}}\right)^{m_1}} + \frac{d \exp\{i(m_1+1)(\pi+\varphi_{11}) + i\pi\varepsilon_{11} m_1\}}{\left(z + R \operatorname{cosec} \frac{\pi}{2(m_1+1)} e^{i\left(\varphi_{11} - \frac{\pi\varepsilon_{11}}{m_1+1}\right)}\right)^{m_1}},$$

i.e. we have obtained those, among the functions considered in (1), for which the disk $|z| \leq R$ is the largest disk of univalence.

Theorem 1 follows from Theorem 2 and Theorem 3.

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CITED LITERATURE

1. Pavel Todorov, DAN, 162, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.