ON THE DENSITY OF FINITE FUNCTIONS IN WEIGHTED SPACES

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Abstract

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MATHEMATICS

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ON THE DENSITY OF FINITE FUNCTIONS IN WEIGHTED SPACES

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1. Let G be an open bounded domain of the n-dimensional Euclidean space E^n; let S be its $(n-1)$-dimensional boundary; let T be an m-dimensional bounded manifold of smoothness 2 (without boundary) ([2]). We shall assume that S is Lipschitz ([3]), and that for $m = n-1$ the manifold T is such that $T \cap G = 0$. In the remaining cases, i.e. for $0 \le m < n-1$, T is situated in E^n in an arbitrary way. By x we shall denote an arbitrary point in E^n.

Let $a = a(t)$ be a nonnegative measurable function, defined for all $t > 0$ and possessing the following property: for every interval $[\delta, A]$, $0 < \delta < A < \infty$, there exist constants c_1 and c_2 such that $0 < c_1 < a(t) < c_2 < +\infty$ for all $t \in [\delta, A]$.

Let g be some closed subdomain of the domain G, all points of which are at a positive distance from the set $T \cup S$. We shall say that $f(x) \in W^1_{p,\alpha}(G,T)$, $1 \le p < \infty$, if $f(x)$ has first generalized derivatives in the sense of S. L. Sobolev ([1]) and if the norm is finite

$$\|f, W^1_{p,\alpha}(G,T)\| = \|\alpha(r)|\operatorname{grad} f|\|_G + \|f\|_g, \tag{1}$$

where $r = r(x)$ is the distance from the point x to T, and $\| \ \|_G$ is the norm in $L_p(G)$. It is easy to see that if $T \cap \overline{G} = 0$, where $\overline{G}$ is the closure of the domain G, then $W^1_{p,\alpha}(G,T)$ coincides with the usual space $W^1_p(G)$ of S. L. Sobolev ([1]). If $m = n-1$, then either all of T lies on S, or part of T is a part of S, and in this case the weight α affects the behavior of the function $f(x)$ only when approaching that part of the boundary S which lies on T. If $0 \le m < n-1$, then all of T, or some part of it, may lie both in G and on S.

If $f(x) \in W^1_{p,\alpha}(G,T)$, then the trace of the function $f(x)$ on $S \setminus T$ exists, while on $T \cap \overline{G}$, depending on α, two cases are possible.

Let

$$\varphi(t) = |\alpha(t)|^{-qt(1-q)(n-m-1)}, \qquad q = p/(p-1).$$

If the integral

$$\int_0^1 \varphi(t)\,dt \tag{2}$$

converges, then the trace of $f(x)$ on $T\cap\overline{G}$ exists; if the integral (2) diverges, then, generally speaking, the trace of $f(x)$ on $T\cap\overline{G}$ does not exist. By $\overset{\circ}{W}{}^1_{p,\alpha}(G,T)$ we shall denote the space of functions $f(x)\in W^1_{p,\alpha}(G,T)$ for which: 1) if the integral (2) converges, then the trace on $S\cup(T\cap\overline{G})$ is equal to zero; 2) if the integral (2) diverges, then the trace on $S\setminus T$ is equal to zero.

By $\overset{\circ}{C}{}^\infty(G,T)$ we shall denote the set of infinitely differentiable functions, each of which is equal to zero outside some closed subdomain of the domain G, all points of which are at a positive dis-

away from $S\cup T$. (This subdomain is, generally speaking, different for each function.)

Theorem 1. The set $\dot{C}^\infty(G,T)$ is dense in the space $\dot{W}^1_{p,\alpha}(G,T)$, $1<p<\infty$.

In the case $\alpha\equiv 1$ this assertion was proved in the paper [3]. In the case when $a(r)$ is a certain power of r and T coincides with all of S, an analogous assertion is found in [4]. In proving Theorem 1 one uses a generalization of Hardy' s inequality [4] and the following lemma.

Lemma. If $f(x)\in\dot{W}^1_{p,\alpha}(G,T)$, then $f(x)$ is such that $\beta(r)f\in L_p(G)$, where

$$\beta(r)=a(r)\varphi(r)\left[\int_r^{2d}\varphi(t)\,dt\right]^{-1},$$

d is the diameter of the domain G.

2. Let

$$L(u)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right),$$

where the functions $a_{ij}=a_{ji}$ $(i,j=1,2,\dots,n)$ are measurable. Suppose there exist constants $c_1>0$ and $c_2>0$ such that for almost all $x\in G$ and all vectors $\xi=(\xi_1,\dots,\xi_n)$ the inequality

$$c_1\alpha|\xi|^2\le\sum_{i,j=1}^n a_{ij}\xi_i\xi_j\le c_2\alpha|\xi|^2,\qquad |\xi|^2=\xi_1^2+\dots+\xi_n^2.$$

holds.

Here we shall assume that the closure of the set $T \cap S$ does not coincide with S.

A function $u(x)$ is said to be a generalized solution of the equation $L(u) = 0$ if $u \in \dot{W}^1_{p,\alpha}(G, T)$ and

$$\int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx = 0$$

for every function $v \in \dot{C}^\infty(G, T)$.

Problem D. Suppose $m < n - 1$, or, if $m = n - 1$, the integral (2) converges. Find a solution of the equation $L(u) = 0$ (classical or generalized) taking prescribed values on S.

Problem E. Suppose T is an $(n-1)$-dimensional manifold and the integral (2) diverges. Find a solution of the equation $L(u) = 0$ taking prescribed values on $S \setminus T$.

Theorem 1 makes it possible to prove the following uniqueness theorem for solutions of Problems D and E.

Theorem 2. The solution (classical or generalized) of Problems D and E in the class of functions $\dot{W}^1_{2,\alpha}(G, T)$ is unique.

An analogous assertion in the case when $\alpha(t) = t^\gamma$, $\gamma \neq 1/2$, was proved in the paper of L. D. Kudryavtsev [2].

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Note: Figure translations are in progress. See original paper for figures.