AN EXACT INTERPOLATION THEOREM

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Abstract

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AN EXACT INTERPOLATION THEOREM

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Let X and Y be linear topological spaces. We shall agree to write $A : X \to Y$ if A is a linear operator acting continuously from X into Y.

The classical Riesz–Thorin theorem (^{1}) asserts, in particular, that if

$$A : L^{p_1} \to L^{q_1}, \qquad A : L^{p_2} \to L^{q_2}, \tag{1}$$

then also

$$A : L^p \to L^q,$$

where $p_1 < p < p_2$ and

$$1/q = 1/q_2 + (1/p - 1/p_2)(1/q_1 - 1/q_2) \times (1/p_1 - 1/p_2)^{-1}.$$

The main question discussed in the present note is the following: does there exist a space X (depending only on p_1, q_1, p_2, q_2, and p) narrower than L^q, and such that, if (1) is fulfilled, then $A : L^p \to X$? It turns out that the answer to this question is, generally speaking, affirmative. Under the additional restrictions $p_1 \leq q_1$, $p_2 \leq q_2$, we describe the minimal space X having this property (see item 2).

1. Let R be a space with a nonnegative countably additive measure μ. For a μ-measurable (complex) function f on R we introduce $f^*(t)$ $(0 \leq t < \infty)$ —the nonincreasing right-continuous function equimeasurable with $|f|$:

$$\text{mes}\{t : f^*(t) > N\} = \mu\{x : |f| > N\} \quad \text{for every } N > 0.$$

In addition, introduce the function $f^{**}(t)$

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

Consider the functional $\|\cdot\|_{\lambda,p}$:

$$\|f\|_{\lambda,p} = \left(\int_0^\infty t^{p\lambda-1} f^{*p}(t)\,dt\right)^{1/p}, \qquad 0 < \lambda < 1, \quad 1 \leq p < \infty,$$

$$\|f\|_{\lambda,\infty} = \sup_{0<t<\infty} t^\lambda f^*(t), \qquad 0 \leq \lambda < 1, \tag{2}$$

and the functional $\langle f\rangle_{\lambda,p}$, which is defined like $\|f\|_{\lambda,p}$, but in the right-hand sides of (2) one must substitute $f^{**}(t)$ in place of $f^*(t)$.

The inequality

$$c_1(\lambda,p)\|f\|_{\lambda,p} \leq \langle f\rangle_{\lambda,p} \leq c_2\|f\|_{\lambda,p}, \qquad 0 < c_1 < c_2 < \infty$$

is valid.

By $L_\mu(\lambda,p)$ we denote $(^{2-4})$ the set of functions f for which $\|f\|_{\lambda,p} < \infty$. $L_\mu(\lambda,p)$ is a Banach space with norm $\langle\cdot\rangle_{\lambda,p}$.

We also introduce the space $L_\mu(1,1)$, coinciding with L^1_μ, setting $\|f\|_{1,1} = \|f\|_{L^1}$. We note that for $\lambda = 1/p$ the space $L_\mu(\lambda,p)$ coincides with L^p_μ.

Theorem 1. *If* $p_1 < p_2$, *then* $L_\mu(\lambda,p_1) \subset L_\mu(\lambda,p_2)$ *and*

$$\|f\|_{\lambda,p_2} \leq C\|f\|_{\lambda,p_1},$$

where $C = C(\lambda,p_1,p_2)$. *In general there are no other inclusion relations between the spaces* $L_\mu(\lambda,p)$.

2. Let X, Y, Z be linear subsets of a linear space V, and suppose that each element $z \in Z$ admits a representation (generally not unique) $z = x + y$, where $x \in X$, $y \in Y$. If T is a linear operator defined on a part of V containing X and Y, then it is also defined on Z.

In this situation we shall say that T is naturally extended to Z. In what follows V will mean the space of all measurable functions.

Theorem 2. *Let* R_1 *and* R_2 *be spaces with measures* ν_1 *and* ν_2, *and let*

$$T:\ L_{\nu_1}(\lambda_i,1) \to L_{\nu_2}(\mu_i,\infty) \qquad (i=1,2, \quad 0 < \lambda_i \leq 1, \quad 0 \leq \mu_i < 1, \quad \lambda_1 < \lambda_2, \quad \mu_1 \neq \mu_2).$$

Then T *is naturally extended to* $L_{\nu_1}(\lambda,p)$, *and*

$$T:\ L_{\nu_1}(\lambda,p) \to L_{\nu_2}(\mu,p),$$

where

$$\lambda_1 < \lambda < \lambda_2, \qquad 1 \leq p \leq \infty,$$

$$\mu = \mu_1 + (\lambda-\lambda_1)(\mu_2-\mu_1)(\lambda_2-\lambda_1)^{-1}. \tag{3}$$

Let us now return to the question posed at the beginning of the note. Let the operator T be defined on simple functions on R_1 and take them into ν_2-measurable functions on R_2. Suppose, moreover, that $T : L^{p_i}_{\nu_1} \to L^{q_i}_{\nu_2}$ $(i = 1, 2)$, where $1 \leq p_i \leq q_i \leq \infty$, $p_i \neq \infty$, $q_i \neq 1$, and $p_1 > p_2$, $q_1 \neq q_2$. By Theorem 1,

$$L_{\nu_1}(1/p_i, 1) \subseteq L_{\nu_1}(1/p_i, p_i) = L^{p_i}_{\nu_1},$$

$$L^{q_i}_{\nu_2} = L_{\nu_2}(1/q_i, q_i) \subseteq L_{\nu_2}(1/q_i, \infty).$$

Therefore the hypotheses of Theorem 2 are satisfied with $\lambda_i = 1/p_i$ and $\mu_i = 1/q_i$, and hence, for $p_1 > p > p_2$, we have $T : L^p_{\nu_1} \to L_{\nu_2}(\mu, p)$, where μ is determined from (3) with $\lambda = 1/p$. Since $p_i \leq q_i$, we have $p \leq 1/\mu$, and $L_{\nu_2}(\mu, p) \subseteq L^{1/\mu}_{\nu_2}$. Thus the range of the operator T lies in the space $L_{\nu_2}(\mu, p)$, generally narrower than the space $L^{1/\mu}$ obtained from the theorem of M. Riesz. As the following theorem shows, the space $L(\mu, p)$ cannot be narrowed further.

We shall agree to say that a measure ν on a space R is nonatomic if, for any two sets E_1 and E_2 such that $E_1 \subset E_2$, $\nu(E_1) < \nu(E_2)$, and for any number s such that $\nu(E_1) < s < \nu(E_2)$, there exists a set E, $E_1 \subset E \subset E_2$, such that $\nu(E) = s$.

Theorem 3. *Let R_1 and R_2 be spaces with nonatomic measures ν_1 and ν_2, $\nu_1(R_1) = \infty$, and let arbitrary collections of numbers $\lambda, \lambda_i, \mu_i, p_i, q_i, p$ $(i = 1, 2)$ be given for which the spaces $L(\lambda_i, p_i)$, $L(\mu_i, q_i)$, $\lambda_1 < \lambda < \lambda_2$, $\mu_1 \neq \mu_2$, $1 \leq p \leq \infty$ make sense, and suppose that for each i (taken separately) one of the following two conditions is satisfied: a) $0 < \lambda_i < 1$, $0 < \mu_i < 1$, $1 \leq p_i \leq q_i \leq \infty$; b) $0 \leq \lambda_i \leq 1$, $p_i = 1$, $q_i = \infty$. Let f be an arbitrary function from the space $L_{\nu_2}(\mu, p)$, where μ is determined from (3). Then there exists an operator T such that*

$$T : \ L_{\nu_1}(\lambda_i, p_i) \to L_{\nu_2}(\mu_i, q_i) \qquad (i = 1, 2)$$

and, consequently, T is naturally extended to $L_{\nu_1}(\lambda, p)$,

$$T : \ L_{\nu_1}(\lambda, p) \to L_{\nu_2}(\mu, p)$$

and $Tg = f$, where $g \in L_{\nu_1}(\lambda, p)$.

Below we give two examples illustrating the application of Theorem 2.

3. Let $\{\varphi_n\}_1^\infty$ be an orthonormal and uniformly bounded system of functions on a space R with measure μ. Consider a space S, whose elements are all positive integers, with measure ν equal to one on each one-point set. If $\{c_n\}_1^\infty$ is a sequence of numbers, then by $\|c\|_{\lambda,p}$ we shall denote the norm in $L_\nu(\lambda, p)$ of the function equal to c_n at the point n.

Theorem 4. 1) Let $f \in L_\mu(\lambda, p)$ $(1/2 < \lambda < 1,\ 1 \leq p \leq \infty)$, and let $\{c_n\}_1^\infty$ be the Fourier coefficients of f with respect to the system $\{\varphi_n\}_1^\infty$. Then

$$\|c\|_{1-\lambda,p} \leq A\|f\|_{\lambda,p}, \qquad A = A(\lambda, p, \{\varphi\}).$$

2) Let, for the sequence $\{c_n\}_1^\infty$, the norm $\|c\|_{\lambda,p}$ be finite $(1/2 < \lambda < 1,\ 1 \leq p \leq \infty)$. Then there exists a function f for which the c_n are the Fourier coefficients with respect to the system $\{\varphi_n\}_1^\infty$, and

$$\|f\|_{1-\lambda,p} \leq A\|c\|_{\lambda,p}, \qquad A = A(\lambda, p, \{\varphi\}).$$

Indeed, let T be the operator assigning to f the function $c(n)$ on the space S, where $c(n) = c_n$, and $\{c_n\}$ are the Fourier coefficients of f. It is clear that $T : L_\mu(1,1) \to L_\nu(0,\infty)$ and $T : L_\mu(1/2,1) \to L_\nu(1/2,\infty)$ (the latter is a consequence of Bessel' s inequality and Theorem 1).

From Theorem 2, 1) now follows; 2) is proved analogously. Theorem 4 contains, as a special case, Paley' s theorem (^{1}).

Theorem 4 is sharp in a certain sense.

Theorem 5. Let $c_1 \geq c_2 \geq c_3 \ldots$ be a sequence tending to zero. In order that the function $f(x) = \sum c_n \cos nx$ belong to $L(\lambda, p)$ $(0 < \lambda < 1,\ 1 \leq p \leq \infty)$, it is necessary and sufficient that $\|c\|_{1-\lambda,p} < \infty$.

This theorem is analogous to the Hardy-Littlewood theorem (^{1}).

4. Consider the operator J:

$$(Jf)(x) = \int_{R_n} \frac{f(t)\,dt}{|t-x|^\alpha} \quad \left(0 < \frac{\alpha}{n} < 1\right), \tag{4}$$

where the integration is over n-dimensional real space R_n. In what follows, spaces of functions defined on R_n with Lebesgue measure are considered.

Theorem 6.

$$J : L(\lambda, p) \to L(\lambda - 1 + \alpha/n, p) \quad (1 - \alpha/n < \lambda < 1,\ 1 \leq p \leq \infty).$$

Theorem 6 is obtained with the aid of Theorem 2 from the fact that

$$J : L(1,1) \to L(\alpha/n, \infty); \qquad J : L(1-\alpha/n, 1) \to L(0,\infty).$$

In particular, according to Theorem 6,

$$J : L^p \to L(1/p - 1 + \alpha/n, p) \subseteq L^r$$

$$(1 - \alpha/n < 1/p < 1,\ 1/r = 1/p - 1 + \alpha/n).$$

Thus Theorem 6 yields a strengthening of the well-known Hardy-Littlewood-Sobolev theorem (^{5,6}).

Theorem 7. Let $f(t)$ in (4) range over all $L(\lambda, p)$ $(1-\alpha/n < \lambda < 1,\ 1 \leq p < \infty)$. Consider all possible measurable functions $g(x)$ such that $|g(x)|$ is majorized by some rearrangement of the function $|(Jf)(x)|$. The set of all such $g(x)$ coincides with $L(\lambda - 1 + \alpha/n, p)$.

Theorem 7 shows that Theorem 6 cannot be strengthened within the class of spaces whose norm is unchanged under rearrangement of functions.

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Note: Figure translations are in progress. See original paper for figures.

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