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# ON QUASICONFORMAL MAPPINGS IN SPACE

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## ON QUASICONFORMAL MAPPINGS IN SPACE

*(Presented by Academician M. A. Lavrent'ev on 14 V 1965)*

In the present note  $Q$ -quasiconformal mappings <sup>(1)</sup> in three-dimensional Euclidean space are considered.

1. Let  $D$  be a ring domain bounded by the sphere  $|X| = 1$  and the segment  $[0, r]$ , and let  $M(D)$  be its modulus. Then <sup>(4)</sup>

$$\ln \frac{1}{r} \leq M(D) \leq \ln \frac{\lambda}{r}, \quad 4 \leq \lambda \leq 12.4 \dots$$

Since  $M(D) = 2\sqrt{\pi} M(\Sigma)$ , where  $\Sigma$  is the family of all possible surfaces  $\sigma$  separating the boundary components of  $D$ , it follows that

$$\frac{1}{2\sqrt{\pi}} \ln \frac{1}{r} \leq M(\Sigma) \leq \frac{1}{2\sqrt{\pi}} \ln \frac{\lambda}{r}.$$

These estimates (with the constant  $\lambda = 18$ ) were obtained earlier by another method by B. V. Shabat.

Hence, as also in <sup>(1)</sup>, it follows:

**Theorem 1.** Let  $Y = f(X)$ ,  $X = (x_1, x_2, x_3)$ ,  $Y = (y_1, y_2, y_3)$ ,  $f(0) = 0$ , be an arbitrary  $Q$ -quasiconformal mapping of the ball  $|X| < 1$  onto itself.

Then at every point of the ball the estimates

$$(|X|/\lambda)^Q \leq |f(X)| \leq \lambda |X|^{1/Q}, \quad 4 \leq \lambda \leq 12.4 \dots \quad (1)$$

hold.

We shall suppose that an extremal  $Q$ -quasiconformal mapping of the ball  $|X| < 1$  onto itself, giving the maximum distortion  $\rho = \rho(r)$ , carries at least one plane section passing through  $[0, r]$  into a plane section passing through  $[0, \rho]$ . Then the distortion under the  $Q$ -quasiconformal mapping  $Y = f(X)$ ,  $f(0) = 0$ , of the

ball  $|X| < 1$  onto itself will not exceed the distortion under the  $Q$ -quasiconformal mapping  $w = \varphi(z)$ ,  $\varphi(0) = 0$ , of the disk  $|z| < 1$  onto itself:

$$|\varphi(z)| \leq \rho,$$

where  $\rho$  is determined from the relation

$$\frac{K'(\rho)}{K(\rho)} = \frac{1}{Q} \frac{K'(r)}{K(r)},$$

in which  $K(r)$  and  $K'(r)$  are, respectively, the complete elliptic integrals of the first kind with moduli  $k = r$  and  $k' = \sqrt{1 - r^2}$  (2), or, as a consequence of this,

$$4^{1-Q}|z|^Q \leq |\varphi(z)| \leq 4^{1-1/Q}|z|^{1/Q}.$$

On the other hand, the author has constructed an example of a  $Q$ -quasiconformal mapping of the ball  $|X| < 1$  onto itself for which

$$\lim_{r \rightarrow 0} \frac{\rho}{r^{1/Q}} = 4^{1-1/Q}.$$

The example is based on a mapping obtained by rotating a plane extremal mapping of the disk onto itself. This shows that the constant  $\lambda$  in (1) cannot be replaced by any smaller one than  $4^{1-1/Q}$ .

Therefore, under the stated assumption, we have

**Theorem 2.** Let  $Y = f(X)$ ,  $f(0) = 0$ , be an arbitrary  $Q$ -quasiconformal mapping of the ball  $|X| < 1$  onto itself.

Then at every point of the ball the estimates hold

$$4^{1-Q}|X|^Q \leq |f(X)| \leq 4^{1-1/Q}|X|^{1/Q},$$

and the constant  $4^{1-1/Q}$  cannot be improved.

2. Let  $Q(D)$  be the lower bound of the numbers  $Q$  for which there exist  $Q$ -quasiconformal mappings of the domain  $D$  onto the ball  $|Y| < 1$ ; if such mappings do not exist, we shall put  $Q(D) = \infty$ . Following Gehring (5), we shall call  $Q(D)$  the **coefficient of quasiconformality** of the domain  $D$ , and any  $Q(D)$ -quasiconformal mapping of the domain  $D$  onto the ball  $|Y| < 1$  an **extremal mapping**.

**Lemma.** Let the function  $w = \varphi(z)$  map  $Q$ -quasiconformally a domain  $G$ , whose boundary contains at least one angular point  $P$  with angle  $\pi\alpha$ ,  $0 \leq \alpha \leq 2$ , onto a domain  $G^*$  with smooth boundary, and suppose there exists a neighborhood  $U$  of the point  $P$  in which

$$|\Delta w/\Delta z| \geq k \quad \text{for } 0 \leq \alpha \leq 1; \quad |\Delta w/\Delta z| \leq 1/k \quad \text{for } 1 < \alpha \leq 2, \quad (2)$$

where  $k$  is some constant.

Then  $Q \geq \max(\alpha, 1/\alpha)$ .

Let  $0 \leq \alpha \leq 1$ , and let, for sufficiently small  $\varepsilon > 0$ ,  $\Gamma_\varepsilon$  be the family of all possible curves joining the boundary components of the annulus  $\varepsilon^2 < |z - P| < \varepsilon$  in  $U$ . Then

$$M(\Gamma_\varepsilon) = \frac{1}{\pi\alpha} \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right). \quad (3)$$

On the other hand, under any mapping with  $Q < 1/\alpha$ , in view of (2) and (3), Theorem 5, the images of the arcs of the circles  $|z - P| = \varepsilon$ ,  $|z - P| = \varepsilon^2$  will lie in the annuli  $k\varepsilon \leq |\varphi(z) - P^*| \leq K\varepsilon$ ,  $k\varepsilon^2 \leq |\varphi(z) - P^*| \leq K\varepsilon^2$ , respectively;  $K = K(Q, k)$ .

Hence

$$M(\Gamma_\varepsilon^*) = \frac{1}{\pi} \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right) \quad (4)$$

and from (3), (4)

$$M(\Gamma_\varepsilon^*) = \alpha M(\Gamma_\varepsilon) + o\left(\ln \frac{1}{\varepsilon}\right),$$

i.e.  $Q \geq 1/\alpha$ .

This contradiction proves the assertion of the lemma. The proof in the case  $1 < \alpha \leq 2$  is analogous.

We shall say that a spatial domain  $D$  contains on its boundary a smooth edge  $AB$  with angle  $\pi\alpha$ ,  $0 \leq \alpha \leq 2$ , if at every point  $P \in AB$  the cross-section normal to  $AB$  has angle  $\pi\alpha$ .

**Theorem 3.** Let the function  $Y = f(X)$  map  $Q$ -quasiconformally a domain  $D$ , containing on its boundary a smooth edge  $AB$  with angle  $\pi\alpha$ ,  $0 \leq \alpha \leq 2$ , onto the ball  $|Y| < 1$ , and suppose that at every point  $P \in AB$  the cross-section normal to  $AB$  is transformed into a smooth surface.

Then  $Q \geq \max(\alpha, 1/\alpha)$ .

The mapping  $f(X)$  can be extended to a homeomorphism of  $D \cup AB$  onto  $\{|Y| < 1\} \cup f(AB)$  <sup>(6)</sup>. Therefore there is a point  $P \in AB$  at which, in the cross-section normal to  $AB$ , (2) is satisfied, and application of the lemma gives the assertion of the theorem.

It follows from this that the assertion of Theorem 3 remains valid if the mapping  $f(X)$  satisfies the smoothness condition only at one of those points  $P \in AB$  at which (2) is fulfilled on the normal sections. In the class of such mappings we have

**Corollary.** *If a domain  $D$  contains on its boundary a smooth edge with angle  $\pi\alpha$ ,  $0 \leq \alpha \leq 2$ , then  $Q(D) \geq \max(\alpha, 1/\alpha)$ , and for  $\alpha = 0$ ,  $Q(D) = \infty$ .*

Let  $(r, \varphi, x_3)$ ,  $(\rho, \theta, y_3)$  be cylindrical coordinates in the spaces  $X$  and  $Y$ , respectively, and let  $D$  be the dihedral angle  $0 < \varphi < \pi\alpha$ ,  $0 < \alpha \leq 2$ . Since the mapping

$$\rho = r, \quad \theta = \varphi/\alpha, \quad y_3 = x_3 \quad (5)$$

transforms  $D$  into a half-space and for it  $Q = \max(\alpha, 1/\alpha)$ , it is extremal by Theorem 3.

Thus, for a dihedral angle  $\pi\alpha$ ,

$$Q(D) = \max(\alpha, 1/\alpha).$$

The extremal mapping (5) is not unique; the mappings

$$\rho = kr, \quad \theta = \varphi/\alpha, \quad y_3 = \varphi(x_3),$$

where  $\varphi(x_3) \in C^1$ ,  $\varphi'(x_3)$  lies between  $k$  and  $k/\alpha$ , and  $k$  is an arbitrary constant, will also be such.

If, in particular,  $D$  is a quarter-space ( $\alpha = 1/2$ ) or the space with the half-plane  $\{x_1 > 0, x_2 = 0\}$  removed ( $\alpha = 2$ ), then  $Q(D) = 2$ .

However, if  $D$  is one-eighth of space (an octant), then the double unfolding is not an extremal mapping. Using orthogonal projection, we see that in this case

$$Q(D) \leq 2 + \sqrt{3} \approx 3.72.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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