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Abstract

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MATHEMATICS

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ON PRIMARY IDEALS IN THE SPACE $\mathcal{L}(0, \infty)$

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We consider the space $\mathcal{L}(0, \infty)$ of functions equal to zero on the left half-line, with norm

$$\|f\| = \int_0^\infty |f(t)|\, dt. \tag{1}$$

This space is a ring with respect to the convolution operation

$$f_1 * f_2 = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau,$$

and the maximal ideals of the ring $\mathcal{L}(0, \infty)$ are identified with the points of the closed half-plane, i.e., each maximal ideal $M(z_0)$ consists of functions $f(t)$ whose Fourier transforms

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) e^{-itz}\, dt \tag{2}$$

vanish at the point z_0 of the half-plane $\operatorname{Im} z \leqslant 0$.

After the appearance of Rudin's work ([1]), a complete description became known of all closed ideals in the ring A of functions analytic in the disk $|z| < 1$ and continuous in the closed disk, with the uniform norm*. Rudin' s results can immediately be reformulated for the ring of functions analytic in the half-plane $\operatorname{Im} z < 0$, continuous in the closed half-plane and having a limit as $|z| \to \infty$, with the uniform norm. Of particular interest is the study of closed ideals in the subring of this ring consisting of functions $F(z)$ for which the representation (2) holds and whose norm is defined by equality (1), or, what is the same thing, in the ring $\mathcal{L}(0, \infty)$, since this study is closely connected with harmonic analysis and synthesis of bounded functions equal to zero on the left half-axis. In the present note we investigate the question of the structure of closed primary ideals of the ring $\mathcal{L}(0,\infty)$, which below we shall call simply primary ideals.

The ring $\mathcal{L}(0,\infty)$ contains no identity; in it there exist ideals contained in no maximal ideal. Slightly modifying the standard terminology for a ring with identity, we shall call such ideals primary ideals corresponding to the point at infinity. By primary ideals corresponding to a point z_0 of the closed lower half-plane we shall mean ideals contained in no maximal ideal $M(z)$, $z \neq z_0$, and in no primary ideal corresponding to the point at infinity.

Three types of primary ideals naturally stand out: primary ideals corresponding to a point of the open half-plane, primary ideals corresponding to the point at infinity, and primary ideals corresponding to a point lying on the real axis.

* Earlier ([2]) G. E. Shilov found all primary ideals of the ring A.

We shall constantly use the fact that a subspace of the convolution ring $\mathcal{L}(0,\infty)$ is a closed ideal if and only if it is invariant under all right shifts $f(t-\tau)$, $\tau > 0$, i.e., together with $f(t)$ it contains all functions $f(t-\tau)$, $\tau > 0$.

Let $\operatorname{Im} z_0 < 0$ and

$$I_k(z_0) = \{f(t) : F^{(j)}(z_0) = 0;\ j = 0, \dots, k\}.$$

It is easy to prove that then to the point z_0 there corresponds a discrete ordered maximal chain of primary ideals $\{I_k(z_0)\}_0^\infty$, ordered in the sense that $I_k(z_0)$ contains $I_m(z_0)$, without coinciding with $I_m(z_0)$, for $k < m$, and maximal in the sense that any primary ideal contained in the maximal ideal $M(z_0) = I_0(z_0)$ coincides with $I_k(z_0)$ for some k.

In the paper ([3]) the following theorem is proved:

In order that the system of right shifts of some family of functions $\{f(t)\}$ from $\mathcal{L}(0,\infty)$ be complete in $\mathcal{L}(0,\infty)$, it is necessary and sufficient that the Fourier transforms of the functions of this family do not vanish simultaneously at any point of the closed half-plane $\operatorname{Im} z \leqslant 0$, and that for no $\gamma > 0$ do all functions of the family vanish simultaneously on the interval $(0,\gamma)$ almost everywhere.

With the help of this theorem one immediately obtains a description of all primary ideals corresponding to the infinitely remote point. Indeed, if one introduces the subspace $I_\gamma(\infty)$ of the space $\mathcal{L}(0,\infty)$ consisting of functions equal to zero on the interval $(0,\gamma)$, then it is easy to see that $I_\gamma(\infty)$ is invariant under right shifts and is not contained in any maximal ideal, i.e. that $I_\gamma(\infty)$ is a primary ideal corresponding to the infinitely remote point. The theorem cited above shows that every primary ideal corresponding to the infinitely remote point and not contained in any maximal ideal coincides with $I_\gamma(\infty)$ for some $\gamma > 0$. Thus, to the infinitely remote point there belongs an ordered maximal continuous chain of primary ideals $\{I_\gamma(\infty)\}_{\gamma>0}$, with

$$\bigcup_{\gamma>0} I_\gamma(\infty) = I_0(\infty) = \mathcal{L}(0,\infty), \quad \bigcap_{\gamma>0} I_\gamma(\infty) = \{0\} * .$$

Let us now pass to the problem of describing all primary ideals corresponding to a point ξ of the real axis. We shall show that in this case, in a certain sense, an analogous situation is preserved.

Let $M(0,\infty)$ be the space dual to the space $\mathcal{L}(0,\infty)$, consisting, as is known, of essentially bounded functions. Denote by $B_{1/2,\alpha}$ the subspace of the space $M(0,\infty)$ consisting of functions that extend to the whole complex plane as entire functions of order $1/2$ and finite type α, and introduce into consideration the sets of functions $I_\alpha(\xi)$ of the space $\mathcal{L}(0,\infty)$

$$I_\alpha(\xi) = \left\{ f(t) : \int_0^\infty f(t) e^{-i\xi t} g(t)\, dt = 0, \ \forall g(t) \in B_{1/2,\alpha} \right\}. \tag{3}$$

The following very simple propositions hold, concerning the sets $I_\alpha(\xi)$ $(\alpha > 0)$:

1°. $I_\alpha(\xi)$ is a subspace of the space $\mathcal{L}(0,\infty)$ invariant under all right shifts; the Fourier transforms of all functions from $I_\alpha(\xi)$ vanish at the point ξ. Consequently, $I_\alpha(\xi)$ is a closed ideal belonging to the maximal ideal $M(\xi)$. We note that $M(\xi) = I_0(\xi)$, since an entire function of zero type and order $1/2$, bounded on the real axis, is constant.

2°. The mapping $f(t) \mapsto f(t)e^{-i\xi t}$ is an isomorphic mapping of the ideal $I_\alpha(\xi)$ onto the ideal $I_\alpha(0)$.

This circumstance makes it possible to restrict ourselves to the case $\xi = 0$.

* The existence of an analogous chain of primary ideals corresponding to the infinitely remote point was first discovered by B. I. Korenblum ([4]) for the normed ring of functions integrable on the whole real axis with exponentially increasing weights.

3°. $f(t) \in I_\alpha(0)$ if and only if $f(\lambda t) \in I_{\alpha\sqrt{\lambda}}(0)$; in particular, $f(\lambda t) \in I_\alpha(0)$ for $\lambda > 1$.

4°. For each point $z_0 \neq 0$ $(\operatorname{Im} z_0 \leq 0)$ and each $I_\alpha(0)$ there exists a function $f(t) \in I_\alpha(0)$ whose Fourier transform is nonzero at this point.

5°. The functions in $I_\alpha(0)$ do not vanish simultaneously almost everywhere on any interval $(0,\gamma)$, $\gamma > 0$.

6°. If $f(t) \in I_\alpha(0)$ for all $\alpha < \alpha_0$, then $f(t) \in I_{\alpha_0}(0)$; in other words,

$$\bigcap_{\alpha<\alpha_0} I_\alpha = I_{\alpha_0}.$$

7°. If $f(t) \in \bigcap_{\alpha>0} I_\alpha$, then $f(t) = 0$ almost everywhere.

8°. $I_\alpha(0)$ is a proper primary ideal.

9°. $I_\alpha(0)$ is contained in $I_\beta(0)$ and does not coincide with $I_\beta(0)$ for $\alpha > \beta$.

The main result of the paper is the following

Theorem 1. *To each point ξ of the real axis there corresponds an ordered maximal continual chain of primary ideals $\{I_\alpha(\xi)\}_0^\infty$, where $I_\alpha(\xi)$ is defined by equality (3). The union of all $I_\alpha(\xi)$ for $\alpha > 0$ coincides with the maximal ideal $M(\xi)$, and the intersection of all $I_\alpha(\xi)$ consists of the identically zero element.*

Equality (3) shows that the annihilator of the primary ideal $I_\alpha(0)$ is the subspace $B_{1/2,\alpha}$ of the space $M(0,\infty)$. We note that the annihilator $I^\perp$ of a closed ideal I, as is known, consisting of functions $g(t) \in M(0,\infty)$ orthogonal to all functions in I, is a subspace of the space $M(0,\infty)$ invariant with respect to left shifts, i.e. $I^\perp$, together with a function $g(t)$, contains functions of the form $g(t+\tau)$, $\tau > 0$. The proof of Theorem 1 is based on the following lemmas.

Lemma 1. *Let $I^\perp$ be the annihilator of some primary ideal I contained in the maximal ideal $M(0)$, and let $g(t) \in I^\perp$. Then $g(t) \in B_{1/2,\alpha}$ for some $\alpha > 0$.*

Lemma 2. *Let I be a primary ideal contained in $M(0)$, and let $g_0(t) \in B_{1/2,\alpha}$, with the type of the function $g_0(t)$ exactly equal to α. Suppose, further, that $g_0(t) \in I^\perp$. Then $B_{1/2,\alpha} \subset I^\perp$.*

Lemma 3. *In order that a function $f(t)$ belong to the primary ideal $I_\alpha(0)$ and not belong to any ideal $I_\beta(0)$, where $\beta > \alpha$, it is necessary and sufficient that its Fourier transform—the function $F(z)$—satisfy the condition*

$$\lim_{y\to 0} y \ln |F(iy)| = 4\alpha^2, \qquad y = \operatorname{Im} z.$$

In conclusion, let us note the following approximation theorems of Wiener and Beurling type, which are easily obtained from the lemmas given above.

Theorem 2. *In order that the system of right shifts of a family $\{f(t)\}$ belonging to the primary ideal $I_\alpha(\xi)$ be complete in this ideal, it is necessary and sufficient that the following conditions hold:*

1. *All functions of the family do not vanish simultaneously on any interval $(0,\gamma)$.*
2. *For each point $z \neq \xi$ $(\operatorname{Im} z \leq 0)$ there is a function of the family whose Fourier transform is nonzero at this point.*
3. $$\inf_{f\in\{f\}} \lim_{y\to 0} y \ln |F(\xi + iy)| = 4\alpha^2.$$

Theorem 3. *Let $g(t) \in B_{1/2,\alpha}$, and let the type of the function $g(t)$ be exactly equal to α. Then by linear combinations of left shifts of the function $g(t)$ one can approximate, in the weak sense, any function from $B_{1/2,\alpha}$.*

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REFERENCES

1. W. Rudin, C. R., **244**, 997 (1957).
2. G. E. Shilov, *Uch. zap. Mosk. univ.*, Ser. Math., **146**, No. 3 (1949).
3. V. P. Gurarii, B. Ya. Levin, *Zap. Kharkov. matem. obshch., Mekh.-mat. fak. Kharkov. gos. univ.*, **30**, 178 (1963).
4. B. I. Korenblyum, *Tr. Moskovsk. matem. obshch.*, **7**, 121 (1958).

Note: Figure translations are in progress. See original paper for figures.