# ON STABLE NORMALLY SOLVABLE EXTENSIONS OF DIFFERENTIAL OPERATORS IN PARTIAL DERIVATIVES

1966

SovietRxiv

## Abstract

## Full Text

UDC 617.946.9+517.43

## MATHEMATICS

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# ON STABLE NORMALLY SOLVABLE EXTENSIONS OF DIFFERENTIAL OPERATORS IN PARTIAL DERIVATIVES

*(Presented by Academician V. I. Smirnov on 29 X 1965)*

In the present work, for a general differential operator in partial derivatives with variable coefficients in a bounded domain, some criteria are established for the stability of the property of normal solvability and of the index of a homogeneous boundary-value problem with respect to small perturbations, in a certain sense, of the boundary conditions. The question of stability with respect to perturbations of the coefficients of the differential expression is also discussed. Analogous questions for correct boundary-value problems (solvable extensions) were considered by us earlier ($^5$).

1. Let

$$\mathcal{P}(x,\mathcal{D}) = \sum_{|\alpha|\le m} a_\alpha(x)\mathcal{D}^\alpha$$

be a differential expression with complex-valued variable coefficients, given in an $n$-dimensional bounded domain $\Omega$, and let $P_0, P$ be the minimal and maximal operators in $L_2(\Omega)$ generated by it. We shall assume that the minimal operators $P_0$ and $\bar{P}_0(\mathcal{P}'(x,\mathcal{D}))$—the differential expression formally adjoint to $\mathcal{P}(x,\mathcal{D})$—have bounded inverses.

Let $\hat{P}(D(P_0) \subset D(\hat{P}) = \hat{D} \subset D(P))$ be a closed normally solvable extension of the operator $P_0$; let

$$\hat{N} = \{\varphi(x),\ \varphi \in \hat{D},\ P\varphi = 0\}, \qquad \hat{M} = \hat{D}(L_2(\Omega)\ominus\hat{N}), \qquad \hat{Z} = L_2(\Omega)\ominus R(\hat{P}).$$

Put

$$c = c(\hat{P}) = \inf_{0\ne\varphi\in\hat{M}} \|P\varphi\|/|\varphi|^*. \tag{1}$$

As is known ($^4$), $0 < c < \infty$. Introduce the dimensions

$$\hat{\alpha} = \dim\hat{N}, \qquad \hat{\beta} = \dim\hat{Z}.$$

If at least one of them is finite, we put

$$\operatorname{ind}\hat{P} = \hat{\alpha} - \hat{\beta}.$$

Consider the Hilbert space $H_P$ of elements $g(x) \in D(P)$ with scalar product

$$[g, h] = (g, h) + (Pg, Ph), \tag{2}$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$. We denote the norm in $H_P$ by $|\cdot|$.

**Lemma 1.** In order that the subspace $\hat{D}(D(P_0) \subset \hat{D} \subset D(P))$ be the domain of definition of a normally solvable extension of the operator $P_0$, it is necessary and sufficient that $H_P$ decompose into the direct sum

$$H_P = \hat{D} \dotplus \hat{V},$$

where $\hat{V}$ is a subspace in $H_P$ satisfying the condition

$$(P\varphi, Pv) = 0, \quad \forall \varphi \in \hat{D},\ \forall v \in \hat{V}.$$

If, in addition, one requires that

$$(\varphi, v) = 0, \quad \forall \varphi \in \hat{N},\ \forall v \in \hat{V},$$

then the subspace $\hat{V}$ is determined uniquely by the extension $\hat{P}$.**

* $\| \cdot \|$ is the norm in $L_2(\Omega)$.

** Another criterion for the normal solvability of the operator $\hat{P}$ $(P_0 \subset \hat{P} \subset P)$ was given by M. I. Vishik $(^3)$.

Together with the subspace $\hat{D}$ defining the normally solvable extension $\hat{P}$, consider another subspace $\tilde{D}(D(P_0) \subset \tilde{D} \subset D(P))$, and let $\tilde{P}$ be the restriction of the operator $P$ to $\tilde{D}$. Let $\tilde{N} = \{\psi(x), P\psi = 0,\ \psi \in \tilde{D}\}$, $\tilde{M} = \tilde{D}(L_2(\Omega) \ominus \tilde{N})$, $\tilde{Z} = L_2(\Omega) \ominus R(\tilde{P})$; $\tilde{\alpha} = \dim \tilde{N}$, $\tilde{\beta} = \dim \tilde{Z}$. Denote by $\chi$ the projection operator projecting $\hat{H}_P$ onto $\hat{V}$ parallel to $\hat{D}$.

**Theorem 1.** Let the operator $\hat{D}$ be normally solvable and let one of the dimensions $\alpha, \beta$ be finite. If

$$|\chi\psi| \leq \delta |\psi|, \qquad \forall \psi \in \tilde{D}, \tag{3}$$

and from the relations

$$\varphi \in \hat{D}, \qquad [\varphi, \psi] = 0, \qquad \forall \psi \in \tilde{D} \tag{4}$$

it follows that $\varphi = 0$, then, under the condition

$$\delta < c(c^2 + 1)^{-1/2}, \tag{5}$$

where $c = c(\hat{P})$ is defined by equality (1), the operator $\tilde{P}$ is also normally solvable. Moreover, if $\hat{\alpha} < \infty$ and $\hat{\beta} < \infty$, then $\operatorname{ind} \tilde{P} = \operatorname{ind} \hat{P}$ and $\tilde{\alpha} \leq \hat{\alpha}$; if

$\hat{\alpha} < \infty$ and $\hat{\beta} = \infty$, then $\tilde{\alpha} \le \hat{\alpha}$ and $\tilde{\beta} = \hat{\beta}$; if $\hat{\alpha} = \infty$ and $\hat{\beta} < \infty$, then $\tilde{\beta} \le \hat{\beta}$ and $\tilde{\alpha} = \hat{\alpha}$.

We outline the proof. For $\hat{\beta} < \infty$, the proof of normal solvability of the operator $\tilde{P}$ is carried out by passing to the adjoint operators $\hat{P}^*$ and $\tilde{P}^*$ and is based on the observation that the gap between their domains of definition in the Hilbert space $H_{\bar{P}}$ with scalar product $[g, h] = (g, h) + (\bar{P}g, \bar{P}h)$ does not exceed $c(c^2+1)^{-1/2}$. In addition, the following is used.

**Lemma 2.** If the operators $P_0^{-1}$ and $\bar{P}_0^{-1}$ are bounded, $\psi_n \in \tilde{M}$, $|\psi_n| = 1$ $(n = 1, 2, ...)$, and $\lim_{n\to\infty} \|P\psi_n\| = 0$, then there exists a subsequence $\{\psi_{n_k}\}$ of the sequence $\{\psi_n\}$ such that

$$\lim_{k\to\infty} [\psi_{n_k}, g] = 0, \qquad \forall g \in H_P.$$

For $\hat{\alpha} < \infty$, normal solvability of $\tilde{P}$ follows from Lemma 1 and the following proposition.

**Lemma 3.** Let the operator $\hat{P}$ be normally solvable, $\hat{\alpha} < \infty$, and

$$\|\chi\psi\| \le \delta|\psi|, \qquad \forall \psi \in \tilde{M}. \tag{6}$$

If $0 < \delta < 1$, then the operator $\tilde{P}$ is also normally solvable.

We note that condition (6) does not imply any estimates for the gap between the subspaces $\hat{D}$ and $\tilde{D}$, and the condition on $\delta$ is less restrictive than (5).

For the proof of the stability of the index and the semistability of the defect numbers, the following considerations are used.

Let $\pi$ be the operator of orthogonal projection of $L_2(\Omega)$ onto $R(\tilde{P})$. Put $R_1 = \pi R(\hat{P})$. Since $\|\pi\hat{r}\| \ge (1 - \delta(1+c^2)^{1/2}c^{-1})\|\hat{r}\|$, $\forall \hat{r} \in R(\hat{P})$, it follows that $R_1$ is a subspace in $L_2(\Omega)$. Let $R_2 = R(\tilde{P}) \ominus R_1$. It is not difficult to verify that the gap in $L_2(\Omega)$ between the subspaces $R(\hat{P})$ and $R_1$ is less than 1. Therefore

$$\hat{\beta} = \dim \hat{Z} = \dim(R_2 \oplus \tilde{Z}) = \dim R_2 + \dim \tilde{Z}. \tag{7}$$

Denote by $\tilde{M}_1$ and $\tilde{M}_2$ subspaces in $\tilde{M}$ such that* $P\tilde{M}_1 = R_1$, $P\tilde{M}_2 = R_2$. Obviously, $\tilde{M} = \tilde{M}_1 + \tilde{M}_2$ and $\dim R_2 = \dim \tilde{M}_2$. The subspace $\tilde{N}_2 = \tilde{N} \oplus \tilde{M}_2$ consists of those and only those elements $\psi \in \tilde{D}$ for which—

* $\tilde{M}_1$ and $\tilde{M}_2$ are closed in $H_P$.

... for which $(I - \chi)\psi \in \hat{N}$. Since $\delta < 1$, it follows from this that if for some $\psi \in \widetilde{N}_2$ we have $[\psi, \varphi] = 0$, $\forall \varphi \in \hat{N}$, then $\psi = 0$. Next it is shown that, for

$\delta < 1$, we have $\hat{D} = (I - \chi)\widetilde{D}$, and from this it is inferred that if for some $\varphi \in \hat{N}$ we have $[\varphi, \psi] = 0,\ \forall \psi \in \widetilde{N}_2$, then $\varphi = 0$. Hence,

$$\hat{\alpha} = \dim \hat{N} = \dim(\widetilde{N} \oplus \widetilde{M}_2) = \dim R_2 + \dim \widetilde{N}. \tag{8}$$

All conclusions of the theorem follow from relations (7), (8).

**Remark.** Theorem 1 admits a partial converse, namely: *if $\hat{P}$ and $\widetilde{P}$ are normally solvable, $\hat{\alpha} < \infty$, $\hat{\beta} < \infty$,* $\operatorname{ind} \widetilde{P} = \operatorname{ind} \hat{P}$, *and conditions (3), (5) are fulfilled, then it follows from relation (4) that* $\varphi = 0$. In proving this assertion one uses a lemma of Bills ($^1$). It can also be shown (in particular on the basis of Lemma 3) that Theorem 3 of paper ($^5$) is a consequence of Theorem 1 proved here; moreover, in the formulation of the sufficient condition of the theorem 3 mentioned, the inequality $\delta < \frac{1}{2}$ may be replaced by the inequality $\delta < 1$.

We shall now establish some conditions for the simultaneous fulfillment of the equalities $\tilde{\alpha} = \hat{\alpha},\ \tilde{\beta} = \hat{\beta}$.

**Lemma 4.** *Let $\hat{P}$ be a normally solvable extension of the operator $P_0$, and let $\tau$ be the orthogonal projector in $H_P$ onto $N$. If $\widetilde{P}$ is also a normally solvable extension of the operator $P_0$, then there exists an $\varepsilon$, $0 < \varepsilon < 1$, such that*

$$\|\tau\psi\| \leqslant \varepsilon|\psi|, \qquad \forall \psi \in \widetilde{M}. \tag{9}$$

**Theorem 2.** *Suppose all the hypotheses of Theorem 1 are fulfilled and that the numbers $\varepsilon, \delta$, entering into conditions (3), (9), are connected by the relation*

$$\varepsilon + \delta < 1. \tag{10}$$

*Then $\hat{\alpha} = \tilde{\alpha},\ \hat{\beta} = \tilde{\beta}$.*

Indeed, it follows from inequality (10) that if $\psi \in \widetilde{M}$ and $(I - \chi)\psi \in \hat{N}$, then $\psi = 0$. But then $\widetilde{N}_2 = \hat{N},\ \widetilde{M}_2 = \{0\},\ R_2 = \{0\}$, so that the proof is completed by reference to relations (7), (8).

2. Denote by $\Gamma$ the boundary of the domain $\Omega$, and let $H(\Gamma)$ be some Hilbert space of vector-functions defined on $\Gamma$, with scalar product $\langle \cdot, \cdot \rangle$. Suppose that for all $u(x), w(x) \in C^\infty(\overline{\Omega})$ the representation

$$(\mathcal{P}u, w) - (u, \mathcal{P}^* w) = \langle \hat{\mathcal{A}} u|_\Gamma, \hat{\mathcal{B}} w|_\Gamma \rangle + \langle \hat{\mathcal{B}}' u|_\Gamma, \hat{\mathcal{A}}' w|_\Gamma \rangle =$$

$$= \langle \widetilde{\mathcal{A}} u|_\Gamma, \widetilde{\mathcal{B}} w|_\Gamma \rangle + \langle \widetilde{\mathcal{B}}' u|_\Gamma, \widetilde{\mathcal{A}}' w|_\Gamma \rangle, \tag{11}$$

is valid, where $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \ldots, \widetilde{\mathcal{A}}'$ are matrix linear differential expressions of order not exceeding $m - 1$ in $C^\infty(\Gamma)$. Let $\hat{D}$ be the closure in $H_P$ of the set of functions $\varphi(x) \in C^\infty(\overline{\Omega})$ for which $\hat{\mathcal{A}}\varphi|_\Gamma = 0$; let $\widetilde{D}$ be the closure in $H_P$ of the set of functions $\psi(x) \in C^\infty(\overline{\Omega})$ for which $\widetilde{\mathcal{A}}\psi|_\Gamma = 0$; and let $\hat{P}$ and $\widetilde{P}$ be the restrictions of the operator $P$ to the subspaces $\hat{D}$ and $\widetilde{D}$, respectively. We shall also suppose that the operator $\hat{P}^*(\widetilde{P}^*)$, adjoint to $\hat{P}(\widetilde{P})$ with respect to the scalar product $(\cdot,\cdot)$, can be defined as the closure in $L_2(\Omega)$ of the differential expression $\mathcal{P}^*(x,\mathcal{D})$, given on functions $w(x) \in C^\infty(\overline{\Omega})$ for which $\hat{\mathcal{A}}'w|_\Gamma = 0$ $(\widetilde{\mathcal{A}}'w|_\Gamma = 0)$.

Suppose that the boundary values of all elements of $\hat{D}, \widetilde{D}, D(\hat{P}^*)$, and $D(\widetilde{P}^*)$ belong to $H(\Gamma)$, and that the differential expressions $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \ldots, \widetilde{\mathcal{A}}'$ can be extended to operators $\hat{A}, \hat{B}, \ldots, \widetilde{A}'$ acting in $H(\Gamma)$ so that

$$\hat{A}\varphi|_\Gamma = 0,\ \forall\varphi \in \hat{D}; \qquad \widetilde{A}\psi|_\Gamma = 0,\ \forall\psi \in \widetilde{D}; \qquad \hat{A}'w|_\Gamma = 0,\ \forall w \in D(\hat{P}^*);$$

$$\widetilde{A}'\widetilde{w}|_\Gamma = 0,\ \forall\widetilde{w} \in D(\widetilde{P}^*); \qquad (P\psi, w) - (\psi, \widetilde{P}w) = \langle B'\psi|_\Gamma, A'w|_\Gamma\rangle;$$

$$\forall\psi \in \widetilde{D}, \quad \forall w \in D(\hat{P}^*); \qquad (P\varphi, \widetilde{w}) - (\varphi, \overline{P}\widetilde{w}) = \langle \widetilde{A}\varphi|_\Gamma,\ \widetilde{B}w|_\Gamma\rangle;$$

$$\forall\varphi \in \hat{D}, \quad \forall\widetilde{w} \in D(\widetilde{P}^*); \qquad (P\varphi, \widetilde{w}) - (\varphi, \overline{P}\widetilde{w}) = \langle \widetilde{B}'\varphi|_\Gamma,\ \widetilde{A}'\widetilde{w}|_\Gamma\rangle;$$

$$\forall\varphi \in \hat{D}, \quad \forall\widetilde{w} \in D(\widetilde{P}^*).$$

**Theorem 3.** Let $\hat{P}$ be a normally solvable extension of the operator ${P_0}^*$ and suppose that for the operator $\widetilde{P}$ the following conditions are satisfied:
1) $\|v\|^2 \le a\langle \hat{A}v|_\Gamma,\ \hat{A}v|_\Gamma\rangle$, $\forall v \in \hat{V}$, representable in the form $v = \psi - \varphi$, where $\psi \in \widetilde{D},\ \varphi \in \hat{D}$;
2) for the same $v$ the estimate

$$|\langle \widetilde{B}'\psi|_\Gamma,\ \hat{A}'Pv|_\Gamma\rangle| \le \eta_1|\varphi|[Pv]$$

holds (here $[h]^2 = \|h\|^2 + \|\overline{P}h\|^2$);
3)

$$|\langle \widetilde{A}\varphi|_\Gamma,\ \widetilde{B}\widetilde{w}|_\Gamma\rangle| \le \eta_2|\varphi|[\widetilde{w}], \quad \forall\varphi \in \hat{D}, \quad \forall\widetilde{w} \in D(\widetilde{P}^*);$$

4)

$$\langle \hat{A}\psi|_\Gamma,\ \hat{A}\psi|_\Gamma\rangle \le \eta_3|\psi|^2, \quad \forall\psi \in \widetilde{D};$$

5) $0 < \eta_2 < 1,\ a\eta_3 + \eta_1^2 < c^2(1+c^2)^{-1}$, where $c = c(\hat{P})$ is determined by equality (1). Then the operator $\widetilde{P}$ is also normally solvable and all the conclusions of Theorem 1 hold.

**Theorem 4.** Let conditions 1)–5) of Theorem 3 be satisfied and

$$|\langle \widetilde{B}'\varphi|_\Gamma,\, \hat{A}'w|_\Gamma\rangle| \leq \varepsilon\|\varphi\|\, |\overline{P}\widetilde{w}|, \quad \forall\varphi\in\hat{N}, \quad \forall\widetilde{w}\in D(\widetilde{P}^*)$$

in such a way that $\overline{P}\widetilde{w}\in\widetilde{M}$. If

$$\varepsilon + (a\eta_3 + \eta_1^2)^{1/2} < 1,$$

then $\tilde{\alpha} = \hat{\alpha}$ and $\tilde{\beta} = \hat{\beta}$.

## 3. Stability

Stability of the property of normal solvability and of the index of an extension with respect to perturbation of the differential expression $\mathcal{P}(x, \mathcal{D})$ by the differential expression $\lambda Q(x, \mathcal{D})$ holds for sufficiently small $|\lambda|$, if (as follows from the general theorems on $\Phi$- and $\Phi_\pm$-operators (4))

$$D(\hat{P}) \subset D(Q).$$

The latter inclusion, as is known, is equivalent to the inequality

$$\|Q\varphi\|^2 \leq C|\varphi|^2, \qquad \forall\varphi\in\hat{D} \quad (C>0 \text{ is a constant}). \tag{12}$$

The question of conditions on the system of boundary operators under which inequality (12) is valid is considered in the works of M. Schechter (7). For arbitrary operators in a Hilbert space with finite and semi-infinite $d$-characteristic, the question of the stability of the property of normal solvability and of the index is considered in the work of Cordes and Labrousse (2)**. The approach developed in the present note makes it possible, in the case under consideration, to avoid the reduction procedure proposed in (2), which reduces the problem to a problem for bounded operators; it gives a less restrictive condition on the size of $\delta$ in Theorem 1 than that which could be obtained by direct application of the results of (2), and permits a reformulation of Theorems 1 and 2 in terms of boundary operators.

*Remark added in proof.* After the submission of the present article for print, the author became aware of works (8,9), containing certain schemes (different from ours) for investigating the stability of the index in Banach and Hilbert spaces. The assertion of our Theorem 1 can also be derived from Theorem 3.4.3 of (9).

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Received
28 X 1965

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* Moreover, at least one of the inequalities $\hat{\alpha} < \infty$, $\hat{\beta} < \infty$ is satisfied.
** We also note the work (6), in which these results are transferred to operators in Banach space.

*Note: Figure translations are in progress. See original paper for figures.*

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