THE GRID METHOD FOR THE NAVIER-STOKES EQUATIONS

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Abstract

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A. KRZYŻWICKI, O. A. LADYZHENSKAYA

THE GRID METHOD FOR THE NAVIER-STOKES EQUATIONS

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We indicate two finite-difference convergent schemes for the initial-boundary value problem in $Q = \Omega \times [0, T]$, where Ω is a bounded three-dimensional domain, $T > 0$, for the system of Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + u^k \frac{\partial \mathbf{u}}{\partial x_k} = -\operatorname{grad} p + \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}\big|_S = 0, \quad \mathbf{u}\big|_{t=0} = \mathbf{a}, \tag{1}$$

where S is the boundary of the domain Ω; $\mathbf{f} = \mathbf{f}(x, t) \in L_2(Q)$, $\mathbf{a} = \mathbf{a}(x) \in L_2(\Omega)$ are given vectors; $\operatorname{div} \mathbf{a} = 0$. Here, as everywhere below, summation from 1 to 3 is understood over the repeated index k.

For plane flows several difference schemes were given (see ($^{4-6}$)). In (^{4}) the convergence of one of them is proved. The schemes proposed by us differ from those previously available.

We divide the entire half-space $E_3 \times [0, \infty)$ into identical parallelepipeds by means of the planes $x_i = k_i h$, $t = k\Delta t$, where $i = 1, 2, 3$, $k = 0, 1, 2, \ldots$, $k_i = 0, \pm 1, \pm 2, \ldots$; h and Δt are some positive numbers.

Denote by Ω_h^k the closed domain on the plane $t = k\Delta t$, consisting of all 3-dimensional elementary cubes $k_i h \le x_i \le (k_i + 1)h$ belonging to Q, and by S_h^k its boundary. By the same symbols we shall also denote the set of nodes (vertices) of the grid belonging to Ω_h^k and S_h^k, respectively. Similarly, by Q_h we denote the set of all elementary parallelepipeds belonging to Q, as well as the set of all grid nodes belonging to Q.

We replace equations (1) by the following "implicit" difference scheme:

$$u^i_{h\bar{t}} - \nu u^i_{hx_k\bar{x}_k} + \frac{1}{2}\, {}^{-0}u^k_h \left(u^i_{hx_k} + u^i_{h\bar{x}_k} \right) = -p_{hx_i} + f^i_h, \tag{2,1}$$

$$u^k_{h\bar{x}_k} = 0, \tag{2,2}$$

$$u_h^i\big|_{S_h} = 0, \tag{2,3}$$

$$u_h^i\big|_{t=0} = a_h^{i(m)}. \tag{2,4}$$

We used the following notation:

$$v_{\bar{t}} = \frac{1}{\Delta t}\,[v(x,t) - v(x,t-\Delta t)],$$

$$v_{x_k} = \frac{1}{h}\,[v(x+e_{kh},t) - v(x,t)],$$

$$v_{\bar{x}_k} = \frac{1}{h}\,[v(x,t) - v(x-e_{kh},t)],$$

$$^{\pm 0}v = v(x, t \pm \Delta t),$$

$$^{\pm k}v = v(x \pm e_{kh}, t),$$

where e_k is the unit vector along the x_k-axis; u_h^i, p_h denote difference analogues of the functions u^i, p; f_h is the averaging of the vector over the cubes of the considered partition of the domain Q. The vectors $\mathbf{a}_h^{(m)}$ are constructed as follows: first we choose a sequence $\mathbf{a}^{(m)}$ converging in $L_2(\Omega)$ to $\mathbf{a}$, of smooth, solenoidal vector fields finite in Ω, and then approximate each vector $\mathbf{a}^{(m)}$ by the difference vector $\mathbf{a}_h^{(m)}$, constructed according to the following rule: if σ is the boundary of the cube $k_ih \le x_i \le (k_i+1)h$, $i = 1,2,3$, and σ_i is that part of it which lies in the plane $x_i = k_ih$, then the value of the component $a_h^{i(m)}$ at the point $x_i = k_ih$, $i = 1,2,3$, is defined by the formula

$$a_h^{i(m)} = h^{-2}\int_{\sigma_i} a^{i(m)}\,d\sigma.$$

Obviously, the vector $\mathbf{a}_h^{(m)}$ is difference-solenoidal, i.e. $a_{hx_i}^{i(m)} = 0$.

Equations (2,1) are considered at all interior points of the grid Ω_h, and equations (2,2) everywhere where none of the four points entering into (2,2) lies outside Ω_h.

It is easy to calculate that the number of unknowns u_h^i, p_h in equations (2,1), (2,2) is equal to the number of these equations. But since among equations (2,2) there is one dependence (the sum of all $u_{hx_i}^i$ is identically zero), we add to our system the equation

$$\sum p_h = 0, \tag{3}$$

where the sum is extended over all those points of Ω_h which are arguments of p_h in equations (2,1).

To prove the unique solvability of the system (2,1)–(2,3), (3) on the layer Ω_h^k, it is enough to verify that the corresponding homogeneous system has only the zero solution, $u_h^i = p_h = 0$. This follows immediately from the identity, valid for any $1 \leq k \leq [T/\Delta t]$:

$$\|u_h(k)\|^2 + 2\nu\Delta t\|u_{hx}(k)\|^2 + \Delta t^2\|u_{ht}(k)\|^2 = \|u_h(k-1)\|^2 + 2\Delta t(f_h, u_h), \tag{4}$$

where

$$\|u_h(k)\|^2 = h^3 \sum u_h^2, \qquad \|u_{hx}(k)\|^2 = h^3 \sum u^i_{hx_l} u^i_{hx_l},$$

$$(f_h, u_h) = \sum f_h^i u_h^i,$$

and the summation in the last formulas is carried out over those grid points of Ω_h^k at which the expressions standing under the summation sign make sense, i.e. in their construction not a single grid point not belonging to Ω_h is used. To obtain (4), it is enough to multiply equation (2,1), taken on the k-th layer, by $2\Delta t h^3 u_h^i(x, k\Delta t)$, and to sum the resulting equalities over all interior points of the grid Ω_h^k.

Theorem 1. The system (2,1)–(2,3), (3) is uniquely solvable on each layer with respect to u_h^i, p_h for arbitrary $\mathbf{f}, \mathbf{a}$.

From identity (4), in a known way, the following basic inequality is derived:

$$\|u_h(k)\|^2 + 2\nu\Delta t \sum_{l=1}^{k} \|u_{hx}(l)\|^2 + \Delta t^2 \sum_{l=1}^{k} \|u_{ht}(l)\|^2 \leq C, \tag{5}$$

which is a difference analogue of the basic energy inequality of hydrodynamics. On the basis of (5), following (1) and using certain completions of the difference functions constructed in (2), we can prove the existence of a subsequence of u_h that converges strongly in $L_2(Q)$, as $h, \Delta t \to 0$, $m \to \infty$, to some u, while h and Δt are chosen consistently···

accordingly. The limiting function $\mathbf{u}$ has generalized derivatives $\partial u/\partial x_k$, equal to the weak limits in $L_2(Q)$ of the sequences u'_{hx_k}. Using this fact, it is easy to prove that $\mathbf{u}$ is a weak solution (in the sense of E. Hopf) of problem (1) (see (3)).

The construction of the solution $\mathbf{u}$ given above is suitable for any number of spatial variables, in particular for the two-dimensional case; and since in this latter case there is a uniqueness theorem for the weak solution, the entire sequence $\mathbf{u}_h$ converges to the solution $\mathbf{u}$. Thus, we have proved

Theorem 2. *From the difference solutions of the mixed problem for the nonstationary Navier—Stokes equations constructed according to the scheme (2,1)–(2,4), one can always extract a subsequence converging to a weak solution (in the sense of E. Hopf) of this problem for any ratio of the steps* h *and* Δt*. Moreover, in the two-dimensional case the entire sequence converges to this solution.*

For solving problem (1) one may also use the following difference scheme

$$u^i_{ht} - \nu u^i_{hx_k\bar{x}k} + \frac{1}{2} u^k_h \left(\overset{+0}{u}{}^{i}_{hx_k} + \overset{+0}{u}{}_{h\bar{x}k} \right) = -\overset{+0}{p}{}_{hx_i} + f^i_h, \tag{6,1}$$

$$u^k_{hx_k} = 0, \tag{6,2}$$

$$u^i_h\big|_{S_h} = 0, \qquad u^i_h\big|_{t=0} = a^i_h(m), \tag{6,3}$$

where

$$v_t = \frac{1}{\Delta t}\left[v(x, t + \Delta t) - v(x, t)\right].$$

Equations (6,1), (6,2) should be considered at the same mesh points as in the right difference scheme. This scheme converges for $\Delta t/h^2 \leq 1/8\nu$.

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Note: Figure translations are in progress. See original paper for figures.