ON F. SEVERI' S CONJECTURE CONCERNING SIMPLY CONNECTED ALGEBRAIC SURFACES

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ON F. SEVERI' S CONJECTURE CONCERNING SIMPLY CONNECTED ALGEBRAIC SURFACES

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1. A nonsingular projective algebraic surface defined over an algebraically closed field k is called **rational** if it is birationally equivalent to the projective plane $\mathbf{P}^2(k)$. The well-known Castelnuovo criterion ([1]) asserts that a surface V is rational if and only if the following birational invariants of it are equal to zero: the irregularity q, the dimension of the vector space of regular differential forms of degree 1; the geometric genus p_g, the dimension of the space of regular differential forms of degree 2; and the bigenus P_2, the dimension of the space of regular double differentials. Suppose now that the field $k = \mathbf{C}$, the field of complex numbers.

F. Severi in ([5]) posed the question whether the conditions $H_1(V, \mathbf{Z}) = 0$ and $p_g = 0$ imply the rationality of the surface. Recall that the condition $H_1(V, \mathbf{Z}) = 0$ splits into two conditions: $q = 0$ and $\operatorname{Tor} H_1(V, \mathbf{Z}) = 0$. This problem may be strengthened by asking whether simply connected algebraic surfaces with $p_g = 0$ are rational. We construct a family of surfaces giving counterexamples to this conjecture.

2. We shall use the results of the works of Ogg and Shafarevich ([4,6]) on the Galois cohomology of Abelian varieties over a function field of constants, which, in the geometric formulation needed by us, are set out in the book ([1]), Ch. VII.

Let a one-dimensional Abelian variety A be given, whose field of constants is the field of rational functions of some nonsingular curve B. The principal homogeneous space over this variety may be interpreted as a surface V with a pencil of elliptic curves, given together with a regular projection $\pi : V \to B$. We shall assume that the fibers of the fibration π contain no exceptional curves of the first kind. In particular, to the variety A there corresponds a surface J, for which there exists a rational section $s : B \to J$. We shall say that $\pi : J \to B$ is the **Jacobian fibration** for the fibration $\pi : V \to B$. Let $C \to B$ be such a normal covering of the base B with Galois group $\mathfrak{G}$ that $V \times_B C \to C$ has a section over C. Let J_C be the nonsingular model of the fibration $J \times_B C$ over C. The sections $\sigma : C \to J_C$ form a group $\mathfrak{U}_J(C)$, which is a $\mathfrak{G}$-operator group. The fibration $\pi : V \to B$ is completely determined by its Jacobian fibration J, the covering $C \to B$, and an element of the cohomology group $u \in H^1(\mathfrak{G}, \mathfrak{U}_J(C))$. Fixing some Jacobian fibration J and taking the inductive limit of the groups $H^1(\mathfrak{G}, \mathfrak{U}_J(C))$ over all possible normal coverings of B, we arrive at the group $\mathfrak{H}(B, J)$. The elements of this group are in one-to-one correspondence with fibrations $\pi : V \to B$ having J as their Jacobian fibration.

3. Our first step toward the construction of counterexamples will consist in taking for J a rational surface with a pencil of elliptic curves. Such surfaces exist and are easily obtained from irreducible pencils of cubic curves in the projective plane. The curve B in this case is rational.

Proposition 1. Any element of the group $\mathfrak{H}(B, J)$ determines an algebraic surface with $q = p_g = 0$.

Proof. Let $u \in \mathfrak{H}(B, J)$ determine a certain surface with a pencil of elliptic curves $\pi : V \to B$. It is known ([1]) that for such surfaces $(K^2) = 0$. Moreover, $\chi(V) = \sum_i \chi(F_i)$, where χ denotes the Euler–Poincaré characteristic, and F_i are the degenerate fibers of the fibration $\pi : V \to B$. Since the characteristics of the degenerate fibers of $\pi : J \to B$ and of the corresponding fibers of the fibration $\pi : V \to B$ are the same ([1]), we obtain $\chi(V) = \chi(J)$. We shall use Noether' s formula and the fact that for rational surfaces $q = p_g = 0$. We have

$$1 - q + p_g = \frac{(K^2) + \chi}{12};$$

since the right-hand sides of this formula coincide for J and V, we obtain $-q(V) + p_g(V) = 0$. Thus the assertion of the proposition will follow from the following lemma.

Lemma 1. A surface $\pi : V \to B$ with a pencil of elliptic curves with rational base B is irregular (i.e. $q > 0$) if and only if the corresponding Jacobian fibration $J \simeq B \times F$, where F is an elliptic curve.

Proof. Assuming that $q > 0$, consider the mapping of V into its Albanese variety ([2]). Using Chou' s theorem ([3]), it is easy to obtain that the Jacobian fibration V has no degenerate fibers. In view of the fact that B is a rational curve, this can occur only in the case when $J \simeq B \times F$ ([1]).

4. It remains to find an element in the group $\mathfrak{H}(B, J)$ such that it determines a simply connected nonrational surface. We note that the condition $H_1(V, \mathbf{Z}) = 0$ entails simple connectedness for our surfaces. Indeed, the fundamental group of a general fiber of a linear pencil of elliptic curves on V, which is evidently commutative, maps epimorphically onto the fundamental group of V. Since $q = 0$ for any surface from $\mathfrak{H}(B, J)$, it remains for us to find a surface for which $\operatorname{Tor} H_1(V, \mathbf{Z}) = 0$. However, by Poincaré duality, for manifolds $\operatorname{Tor} H_1(V, \mathbf{Z}) = \operatorname{Tor} H^3(V, \mathbf{Z}) \simeq \operatorname{Tor} H_2(V, \mathbf{Z})$. Every two-dimensional cycle with torsion is algebraic ([2]) and $\dim \operatorname{Pic}(V) = q = 0$, where $\operatorname{Pic}(V)$ is the Picard variety of the surface V ([2]). Hence we obtain

that $\operatorname{Tor} H_1(V, \mathbf{Z}) \simeq \operatorname{Tor} C(V)$, where $C(V)$ denotes the group of divisor classes on V.

Lemma 2. Let V be a surface with a pencil of elliptic curves with $q = p_g = 0$. Any divisor $D \in \operatorname{Tor} C(V)$ is linearly equivalent to a divisor contained in the fibers of V.

Proof. Let $D \in \operatorname{Tor} C(V)$. We have $D \nsim 0$, but $nD \sim 0$, where n is some integer. It follows that the linear system $|D|$ is empty, and hence $l(D) = \dim |D| + 1 = 0$. Using the Riemann-Roch inequality, we obtain

$$l(K - D) \geq D(K - D)/2 + 1 - q + p_g = 1.$$

Thus there exists a divisor $D' \geq 0$ such that $K - D \sim D'$. Since for any fiber F one has $(K \cdot F) = 0$ and $(K \cdot D) = 0$, D' does not intersect any fiber and, by virtue of its nonnegativity, is contained in the fibers of V. As is known ([1]), K contains a divisor belonging to the fibers of V. Hence, since $D \sim K - D'$, we obtain the assertion we need.

One of the consequences of the main results of the works of Ogg and Shafarevich are the following assertions:

A. There is a homomorphism for any B and J:

$$\varphi : \mathfrak{H}(B, J) \to \sum_{b \in B} \operatorname{Char} H_1(U_b),$$

where $\operatorname{Char} H_1(U_b)$ is the group of periodic characters of the one-dimensional homology group of the fiber U_b with integer coefficients.

B. Let $V \in \mathfrak{H}(B, J)$. Denote by $h_b(V)$ the image of V under the homomorphism $\varphi_b : \mathfrak{H}(B, J) \to \operatorname{Char} H_1(U_b)$. Then $h_b(V)$ is nonzero only in the case when the fiber V over the point b is a multiple fiber. The multiplicity of the fiber V at the point b is equal to the order of $h_b(V)$ in the group $\operatorname{Char} H(U_b)$, which is an infinitely divisible group.

C. If J has at least one degenerate fiber, then the homomorphism φ is an epimorphism.

Let S be a finite set of points of the curve B over which there lie degenerate fibers U_b with $H_1(U_b) = 0$. Suppose that J has at least one degenerate fiber (this will be so, for example, if J is rational). Thus, from results A, B, C it follows that in the group $\mathfrak{H}(B, J)$ there always exist surfaces having, at prescribed points not belonging to S, multiple fibers with prescribed multiplicities.

Theorem. *Let J be a rational surface. Let $u \in \mathfrak{H}(B, J)$ represent a surface with two multiple fibers of multiplicities l_1 and l_2, where $l_1 \neq l_2$ are prime numbers. The element u defines a nonrational simply connected algebraic surface with $p_g = 0$.*

Proof. First of all let us show that such a surface is nonrational. Suppose that u defines a rational surface $\pi : V \to B$ with a pencil of elliptic curves. V is not a minimal model for rational surfaces, since for the latter $(K^2) = 8$ or 9 (1), whereas on V $(K^2) = 0$. Let S be an exceptional curve of genus I on V. By the assumption made at the very beginning of the paper, S is not contained in the fibers of the fibration π. Therefore $(S \cdot F) = m > 0$, where F is a variable fiber of π. Obviously, the multiplicity of any fiber divides the number m. We shall show that on V there exists a fiber of multiplicity m. Using the Riemann-Roch inequality for the divisor $-K$, we obtain

$$l(-K) + l(2K) \geq (K^2) + 1 - q + p_g.$$

Since V is rational, $P_2 = l(2K) = q = p_g = 0$, i.e. $l(-K) \geq 1$.

Let $D \in |-K|$. The divisor D is a rational combination of the fibers of the fibration π (1), i.e. $D = \sum_i \frac{n_i}{m_i} F_i$, where m_i is the multiplicity of the fiber F_i, $n_i \geq 0$. Since S is an exceptional curve of genus I, $(S \cdot K) = -(S \cdot D) = -1$. Thus, $\frac{1}{m} = \sum_i \frac{n_i}{m_i}$. Taking into account that all m_i divide m, we obtain that there exists a fiber F_i for which $m_i = m$. Thus, on rational surfaces with a pencil of elliptic curves the multiplicities of all fibers divide the multiplicity of some fiber. (In fact, one can even show that on such surfaces there is only one multiple fiber.) Thus the element u cannot represent a rational surface.

It remains to show, by virtue of Proposition 1 and Lemma 2, that in the fibers of the fibration $\pi : V \to B$ which defines the element u, there are no divisors with twisting. Suppose that D is such a divisor. Since $(D^2) = 0$, Lemma 1 from § 2, Ch. VII of the book (1) shows that $D \sim mF + kC_0 + rC_1$, where m, k, r are integers, and C_0 and C_1 are such curves on V that the multiple fibers F_0 and F_1 of V are equal respectively to $l_1 C_0$ and $l_2 C_1$. We may assume that $|k| < l_1$ and $|r| < l_2$. We have

$$nD \sim nmF + nkC_0 + nrC_1 \sim 0. \tag{1}$$

Multiplying equality (1) first by l_1, and then by l_2, we obtain that $l_1 l_2 / n$ and it must be

$$l_1 l_2 m + k l_2 + r l_1 = 0. \tag{2}$$

An elementary argument shows that equation (2), for integers m, k, r satisfying $|k| < l_1$, $|r| < l_2$, has no solution. The contradiction obtained proves the theorem.

5. It would be very interesting to know the answer to the following question: do all surfaces for which Severi' s conjecture is false belong to the family of surfaces with a pencil of elliptic curves?

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Note: Figure translations are in progress. See original paper for figures.