ON A METHOD FOR INVESTIGATING NONLINEAR SINGULAR EQUATIONS

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Abstract

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MATHEMATICS

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ON A METHOD FOR INVESTIGATING NON-LINEAR SINGULAR EQUATIONS

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In the present note a new method is proposed for investigating the equation

$$u(x) = \lambda \int_a^b \frac{f[x, s, u(s)]}{s - x} \, ds \tag{1}$$

in the Guseinov class $H^K_{\alpha,\beta,\delta}$ (1).

We first give several functional inequalities for functions from the Hölder class H, δ and $H^K_{\alpha,\beta,\delta}$ $(0 < \alpha + \delta,\ \beta + \delta,\ \delta,\ \alpha,\ \beta < 1)$, which were obtained by one of the authors of the present note.

1°. Let $u(x) \in H^K_{\alpha,\beta,\delta}$. Then $W(x) = u(x)\psi(x)$ belongs to the class $H^0_{l_1,K,\delta}$, where $\psi(x) = (x-a)^{\alpha+\delta}(b-x)^{\beta+\delta}$, and $H^0_{l_1,K,\delta}$ is the Hölder class of functions that vanish at the endpoints of the interval (a, b),

$$l_1 = \left(\frac{5}{4}\right)^{\alpha+\delta} + 4^{-\alpha}\left(\frac{5}{4}\right)^{\beta+\delta}(b-a)^\delta + 4^{-\beta}(b-a)^\delta.$$

2°. If $u(x) \in H^0_{K,\delta}$, then $W(x) = u(x)(x-a)^{-\alpha-\delta}(b-x)^{-\beta-\delta}$ belongs to the class $H^{l_2 K}_{\alpha,\beta,\delta}$; l_2 is a constant independent of K.

3°. If $u(x) \in H_{K,\delta}$, then

$$\max_{a \le x \le b} |u(x)| \le l K^{1/(1+\delta p)} \left\{ \int_a^b |u(x)|^p \, dx \right\}^{\delta/(1+\delta p)}, \qquad p > 0;$$

$$l = \begin{cases} \dfrac{(b-a)^{p\delta^2/(1+\delta p)}}{2^\delta} - \dfrac{\sqrt[p]{2}}{(b-a)^{\delta/(1+\delta p)}}, & \text{if } b - a < \dfrac{2^{(1+\delta p)/\delta p}}{(\delta p)^{1/\delta}} = \delta_0, \\ (\delta p)^{1/(1+\delta p)} + \left(\dfrac{1}{\delta p}\right)^{\delta p/(1+\delta p)}, & \text{if } b - a \ge \delta_0. \end{cases}$$

From 1° and 3° it follows:

4°. If $u(x) \in H^K_{\alpha,\beta,\delta}$, then

$$\sup_{a<x<b} \{|u(x)|\psi(x)\} \leq l(l_1 K)^{1/(1+\delta)} \left\{ \int_a^b \rho(x)|u(x)|^p \, dx \right\}^{\delta/(1+\delta p)},$$

where $\rho(x) = [\psi(x)]^p$.

Since $\rho(x) \leq (b-a)^{\bar{\gamma}} \rho_1(x)$, $\rho_1(x) = (x-a)^{\gamma}(b-x)^{\gamma'}$, $\bar{\gamma} = (\alpha+\delta)p + (\beta+\delta)p - \gamma - \gamma'$, $\varphi - 1 < \gamma \leq \alpha p + \delta p$, $\beta p - 1 < \gamma' \leq \beta p + \delta p$, from 4° we obtain:

5°. If $u(x) \in H^K_{\alpha,\beta,\delta}$, then

$$\sup_{a<x<b} \{|u(x)|\psi(x)\} \leq l_3 l(l,K)^{1/(1+\delta p)} \left\{ \int_a^b \rho_1(x)|u(x)|^p \, dx \right\}^{\delta/(1+\delta p)},$$

$$l_3 = (b-a)^{\gamma\delta/(1+\delta p)}.$$

6°. If $u(x) \in H_{K,\delta}$, then

$$\sup_{a \leq x,\, y \leq b} \frac{|u(x) - u(y)|}{|x-y|^{\delta'}} \leq 2^{1-\delta'/\delta} K^{\delta'/\delta} \{\max |u(x)|\}^{(\delta-\delta')/\delta}, \qquad 0 < \delta' < \delta.$$

From 1° and 6° it follows

7°. For $u(x) \in H^K_{\alpha,\beta,\delta}$ one has

$$\sup_{a<x,\, y<b} \left\{ \frac{|u(x)\psi(x) - u(y)\psi(y)|}{|x-y|^{\delta'}} \right\} \leq$$

$$\leq 2^{(\delta-\delta')/\delta} (l,K)^{\delta'/\delta} \left\{ \sup_{a<x<b} |u(x)|\psi(x) \right\}^{(\delta-\delta')/\delta}.$$

8°. If $u(x) \in H^K_{\alpha,\beta,\delta}$, then

$$\sup_{\substack{a<x,\, y<b \\ |y-x| \leq \sigma(x)}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^{\delta'}} (x-a)^{\alpha+\delta} (b-y)^{\beta+\delta} \right\} \leq$$

$$\leq l_4 K^{\delta'/\delta} \left\{ \sup_{a<x<b} |u(x)|\psi(x) \right\}^{(\delta-\delta')/\delta},$$

$$l_4 = \left\{(4/3)^{\alpha+\delta} + (5/4)^{\beta+\delta}\right\}^{(\delta-\delta')/\delta}, \qquad \min\left\{\frac{x-a}{4}, \frac{b-x}{4}\right\} = \sigma(x).$$

From inequalities 5° and 8° it follows that

$$\|u\|_1 \le \tilde{f}_1\left(\|u\|_{L_p(\rho_1)}\right), \tag{*}$$

where

$$\tilde{f}_1(x) = l_3 l(l,K)^{1/(1+\delta p)} x^{\delta p/(1+\delta p)} + l_4\{l_3 l(l,K)^{1/(1+\delta p)}\}^{(\delta-\delta')/\delta} x^{p(\delta-\delta')/(1+\delta p)},$$

$$\|u\|_1 = \sup_{a<x<b}\{|u(x)|\psi(x)\} +$$

$$+ \sup_{\substack{a<x<b \\ |\Delta x|<\sigma(x)}} \left\{\frac{|u(x+\Delta x) - u(x)|}{|\Delta x|^{\delta'}}(x-a)^{\alpha+\delta}(b-x-\Delta x)^{\beta+\delta}\right\},$$

$$\|u\|_{L_p(\rho_1)} = \left\{\int_a^b \rho_1(x)|u(x)|^p\, dx\right\}^{1/p}.$$

From inequalities 5° and 7° it follows that

$$\|u\|_2 \le f_2\left(\|u\|_{L_p(\rho_1)}\right), \tag{**}$$

where

$$f_1(x) = l_3 l(l,K)^{1/(1+\delta p)} x^{\delta p/(1+\delta p)} + 2^{(\delta-\delta')/\delta}(l,K)^{\delta'/\delta} \times$$

$$\times \{l_3 l(l,K)^{1/(1+\delta p)}\}^{(\delta-\delta')/\delta} x^{p(\delta-\delta')/(1+\delta p)},$$

$$\|u\|_2 = \sup_{a<x<b}\{|u(x)|\psi(x)\} +$$

$$+ \sup_{\substack{a<x<b \\ |\Delta x|<\sigma(x)}} \left\{\frac{|u(x+\Delta x)\psi(x+\Delta x) - u(x)\psi(x)|}{|\Delta x|^{\delta'}}\right\}.$$

Theorem 1. Let $f(x,s,u)$ be defined for $a < s,\ s < b,\ -\infty < u < +\infty$ and satisfy the conditions

$$|f(x+\Delta x, s+\Delta s, u+\Delta u) - f(x,s,u)| \le \frac{A\{|\Delta x|^{\delta'} + |\Delta s|^{\delta}\}}{(s-a)^{\alpha+\delta}(b-s-\Delta s)^{\beta+\delta}} + B|\Delta u|, \tag{2}$$

$$\int_a^b \frac{f(x,s,0)}{s-x}\, ds \in H^{K_0}_{\alpha,\beta,\delta}, \qquad 0 < \delta < \delta_1 \le 1. \tag{3}$$

Suppose, further, that $F(x,s,u) = f(x,s,u) - f(s,s,u)$ satisfies the condition

$$|F(x,s,u) - F(x,s,v)| \le g(x,s)|u-v|, \tag{4}$$

where $g(x,s)$ is a nonnegative function such that the integral

$$R_1 = \left\{ \int_a^b \rho_1(x) \left[\int_a^b \rho_1^{-q/p}(s) \left| \frac{g(x,s)}{s-x} \right|^q ds \right]^{1/p} \right\} \tag{5}$$

converges for some $p > 1$, $1/p + 1/q = 1$, $\gamma < 1$, $\gamma' < 1$. Then the operator

$$Mu = \lambda \int_a^b \frac{f[x,s,u(s)]}{s-x}\, ds$$

for small λ maps $H^K_{\alpha,\beta,\delta}$ into $H^K_{\alpha,\beta,\delta}$ and is a contraction operator in the space $H^K_{\alpha,\beta,\delta}$ with metric

$$\rho_2(u,v) = \|u-v\|_{L_p(\rho_1)}$$

provided

$$|\lambda| < \lambda_0 = \min\{K/K', 1/R_2\}, \qquad K' = K_0 + C_0(2A + KB),$$

$$R_2 = C_1 B + R_1, \quad C_0 \text{ and } C_1 \text{ are certain constants.}$$

If one takes into account inequality $(*)$, then we have proved

Theorem 2. If $f(x,s,u)$ satisfies the conditions of Theorem 1 and $|\lambda| < \lambda_0$, then equation (1) has a unique solution $u_0^*(x) \in H^{\alpha K}_{\alpha,\beta,\delta}$, and this solution can be found by the method of successive Picard approximations

$$u_n(x) = \lambda \int_a^b \frac{f[x,s,u_{n-1}(s)]}{s-x}\, ds, \qquad u_0(x) \in H^K_{\alpha,\beta,\delta}.$$

The successive approximations converge in the sense of the metric

$$\rho_3(u,v) = \|u-v\|_2,$$

and

$$\rho_3(u_n, u_0^*) \le f_1 \left\{ \frac{R_3^n}{1-R_3} \rho_2(Mu_0, u_0) \right\}, \tag{6}$$

where $R_3 = |\lambda| R_2 < 1$.

If one takes into account that

$$\rho_2(Mu_0,u_0) \le KR_4, \quad R_4 = 2\int_a^b \tilde{\rho}_1(x)\,dx, \qquad \tilde{\rho}_1(x) = (x-a)^{\gamma-\alpha p}(b-x)^{\gamma'-\beta p},$$

then, substituting its expression in place of $f_1(x)$, we obtain

$$\rho_3(u_n,u_0^*) \le K\{\tilde{l}_4 R_3^{n\delta p/(1+\delta p)} + l_5 R_3^{np(\delta-\delta')/(1+\delta p)}\}, \tag{6'}$$

where

$$\tilde{l}_4 = l_3 l_1 \left(\frac{R_4}{1-R_3} \right)^{\delta p/(1+\delta p)},$$

$$l_5 = 2^{(\delta-\delta')/\delta} l_1^{(1+\delta' p)/(1+\delta p)} l_3^{(\delta-\delta')/\delta} \left(\frac{R_4}{1-R_3} \right)^{p(\delta-\delta')/(1+\delta p)}.$$

Inequalities of type $(6')$ can also be obtained in the sense of the metric

$$\rho_4(u,v) = \|u-v\|_1.$$

It should be noted that inequality $(6')$ for nonlinear singular equations has been obtained here for the first time.

On the basis of the inequalities given above, a theorem is proved which characterizes the dependence of the solution of equation (1) on the parameter λ.

Theorem 3. Suppose that $f(x,s,u)$ satisfies all the conditions of Theorem 2. Suppose, further, that $u(x,\lambda_1)$, $u(x,\lambda_2)$ are solutions of equation (1), corresponding to the parameters λ_1, λ_2 $(|\lambda_1| < \lambda_0,\ |\lambda_2| < \lambda_0)$. Then

$$\sup_{a<x<b}\{|u(x,\lambda_1)-u(x,\lambda_2)|\psi(x)\}\le R_6|\lambda_1-\lambda_2|^{\delta p/(1+\delta p)},$$

$$R_6 = l(2l,K)^{1/(1+\delta p)}\left(\frac{R_5}{1-R_2|\lambda_2|}\right)^{\delta p/(1+\delta p)}, \tag{7}$$

$$R_5^p = \int_a^b \rho_1(x)\left\{\int_a^b \frac{f[x,s,u(s,\lambda_1)]}{s-x}\,ds\right\}^p dx.$$

If in inequality (7) we set $p = 1/(1-\delta)$, taking $\gamma = \alpha p - 1 + \varepsilon$, $\gamma' = \beta p - 1 + \varepsilon_1$, $0 < \varepsilon, \varepsilon_1 < 1$, we obtain

$$\sup_{a<x<b}\{|u(x,\lambda_1)-u(x,\lambda_2)|\psi(x)\}\le R_6|\lambda_1-\lambda_2|^{\delta}$$

under the condition that integral (5) converges.

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1. A. I. Guseinov, *Izv. AN SSSR, ser. matem.*, **12**, 193 (1948).

Note: Figure translations are in progress. See original paper for figures.