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ON THE HILBERT PROBLEM FOR A HOLOMORPHIC VECTOR

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In this note we consider the problem of linear conjugation (the Hilbert problem) for a holomorphic vector (briefly, problem H). We shall show that, under certain conditions on the conjugation matrix G, problem H is Fredholm. In addition, by introducing the adjoint problem H′ into consideration, we obtain a necessary and sufficient condition for the solvability of problem H. We note that the Hilbert problem for a holomorphic vector was first considered by A. V. Bitsadze ([1]) in the case when the constant matrix G (see formula (2) below) has the form

$$G_0 = \begin{Vmatrix} g_1 & g_2 & g_3 & g_4 \\ -g_2 & g_1 & -g_4 & g_3 \\ -g_3 & g_4 & g_1 & -g_2 \\ -g_4 & -g_3 & g_2 & g_1 \end{Vmatrix}. \tag{1}$$

We investigate problem H by means of singular integral equations, following the ideas of Chapter IV of the book by I. N. Vekua ([4]). The adjoint problem H′ is introduced analogously to the way this was done by B. V. Boyarskii ([3]) in the planar case.

A vector $U(x)$ satisfying a certain elliptic system $DU = 0$ (see ([2])) is called **holomorphic**. A vector $V(x)$ satisfying the system $D'V = 0$ (the prime denotes transposition) will be called **co-holomorphic**.

Let S be a Lyapunov surface, homeomorphic to a sphere, lying in Euclidean space E_3. Consider problem H (see ([1])): to find a piecewise-holomorphic vector $U(x)$ satisfying the boundary condition on S:

$$U^+(y) = G(y)U^-(y) + f(y), \tag{2}$$

where $G(y), f(y) \in C_\alpha(S)$, $0 < \alpha < 1$.

Denote by M_{ij}^{kl} the minor composed of the elements of the k-th and l-th rows and the i-th and j-th columns of the matrix $G(y)$, and let

$$
\begin{aligned}
\Gamma_{11} &= M_{13}^{13} + M_{13}^{42} + M_{42}^{42} + M_{42}^{13} + M_{14}^{23} + M_{14}^{14} + M_{23}^{23} + M_{23}^{14},\\
\Gamma_{22} &= M_{12}^{12} + M_{12}^{34} + M_{34}^{12} + M_{34}^{34} + M_{14}^{23} + M_{14}^{14} + M_{23}^{23} + M_{23}^{14},\\
\Gamma_{33} &= M_{12}^{12} + M_{12}^{34} + M_{34}^{12} + M_{34}^{34} + M_{13}^{42} + M_{13}^{13} + M_{42}^{42} + M_{42}^{13},\\
2\Gamma_{12} = 2\Gamma_{21} &= -(M_{12}^{13} + M_{12}^{42} + M_{34}^{13} + M_{34}^{42} + M_{13}^{12} + M_{13}^{34} + M_{42}^{12} + M_{42}^{34}),\\
2\Gamma_{13} = 2\Gamma_{31} &= -(M_{12}^{14} + M_{12}^{23} + M_{34}^{14} + M_{34}^{23} + M_{14}^{12} + M_{14}^{34} + M_{23}^{12} + M_{23}^{34}),\\
2\Gamma_{23} = 2\Gamma_{32} &= -(M_{13}^{14} + M_{13}^{23} + M_{42}^{14} + M_{42}^{23} + M_{14}^{13} + M_{14}^{42} + M_{23}^{42} + M_{23}^{13}).
\end{aligned}
$$

Everywhere in this note we shall assume that the following condition Γ is fulfilled: all principal minors (including the determinant) of the matrix with elements Γ_{ij} $(i, j = 1, 2, 3)$ are positive on S. We note that for matrices of the form (1) condition Γ coincides with the condition $\det G(y) \neq 0$.

With the help of an integral of Cauchy type (see (2)), problem H is reduced to an equivalent system of singular integral equations for the unknown vector $\mu(y)$:

$$L\mu \equiv (G + E)\mu + (E - G)P\mu = 2f, \tag{3}$$

where

$$P\mu = \frac{1}{2\pi} \iint\limits_S D' \frac{1}{|y - \xi|} DS_\xi \mu(\xi), \qquad y \in S, \qquad DS_y = \left(\sum_{i=1}^{3} \alpha_i \gamma_i \right) ds_y^*,$$

α_i are the direction cosines of the exterior normal to S at the point y, and E is the identity matrix of order four.

We shall consider equation (3) in the space $L_p(S)$ for some $p > 1$. Let $\sigma(L)$ be the symbolic matrix of the operator L. A direct calculation shows that

$$\operatorname{Re} \det \sigma(L) = 4 \sum_{i,j=1}^{3} \Gamma_{ij} \tau_i \tau_j, \tag{4}$$

where (τ_1, τ_2, τ_3) is a unit tangent vector to the surface S at the point y. From condition Γ it follows that the quadratic form (4) is positive definite, so that

$$\det \sigma(L) \neq 0. \tag{5}$$

From the results of S. G. Mikhlin (see ([5]), § 40) it follows that the operator L is Noetherian in the space $L_p(S)$, i.e., first, the homogeneous equation (3) and the adjoint homogeneous equation

$$L^*\chi \equiv (G' + E)\chi + P^*(E - G')\chi = 0 \qquad (6)$$

have only finite numbers k and k' of linearly independent solutions and, second, for the solvability of equation (3) it is necessary and sufficient that

$$\iint_S \chi_j'(y) f(y)\, ds_y = 0 \qquad (j = 1, 2, \dots, k'), \qquad (7)$$

where $\chi_1, \dots, \chi_{k'}$ is a complete system of linearly independent solutions of equation (6). By virtue of condition (5), any solution from the space $L_q(S)$, $q > 1$, of equations (3) and (6) belongs to $C_\alpha(S)$ (see ([7])); thus the operator L is Noetherian also in the space $C_\alpha(S)$.

Let us introduce the adjoint homogeneous problem H_0': to find a piecewise-cogolomorphic vector $V(x)$ satisfying the boundary condition on S:

$$G^*(y) V^+(y) = V^-(y), \qquad (8)$$

where

$$G^*(y) = \left(\sum_{i=1}^{s} \alpha_i \gamma_i\right) G'(y) \left(\sum_{i=1}^{3} \alpha_i \gamma_i\right).$$

With the help of an integral of Cauchy type for a cogolomorphic vector, problem H_0' is reduced to an equivalent system of singular integral equations:

$$M\nu \equiv (G^* + E)\nu + (G^* - E)Q\nu = 0, \qquad (9)$$

where

$$Q\nu = \frac{1}{2\pi} \iint_S D \frac{1}{|y - \xi|} D' S_\xi \nu(\xi).$$

Solutions of equations (6) and (9) can be connected by the relation

$$\nu = \left(\sum_{i=1}^{3} \alpha_i \gamma_i\right) (G' - E)\chi. \qquad (10)$$

* γ_i $(i = 1, 2, 3)$ are constant matrices of order four, with the help of which the operator $D = \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x_i}$ is forr

From formulas (10) and (6) we obtain

$$2\chi = -\left(\sum_{i=1}^{3} a_i'\gamma_i\right)(v + Qv). \tag{11}$$

Therefore the problem H_0' has exactly k' linearly independent solutions $V_1(x), \dots, V_{k'}(x)$. From formulas (7) and (11) it follows:

Theorem 1. *For the solvability of the nonhomogeneous boundary-value problem* H*, it is necessary and sufficient that, for every solution* $V_j(x)$ $(j = 1, \dots, k')$ *of the adjoint homogeneous problem* H_0'*, the condition*

$$\iint_S \{v_j^+(y)\}' \, DS_y f(y) = 0 \tag{12}$$

be satisfied.

The integer $\varkappa(G) = k - k'$ will be called the **index** of the boundary-value problem H generated by the matrix G.

Consider the problem H generated by the matrix $G_t(y)$, depending on the parameter t, $0 \le t \le 1$,

$$G_t(y) = tG(y) + (1-t)G_0(y),$$

where $G_0(y)$ is a matrix of the form (1) with elements

$$\begin{aligned} g_1 &= \tfrac{1}{4}(g_{11} + g_{22} + g_{33} + g_{44}), \\ g_2 &= \tfrac{1}{4}(g_{12} - g_{21} + g_{43} - g_{34}), \\ g_3 &= \tfrac{1}{4}(g_{13} - g_{31} + g_{24} - g_{42}), \\ g_4 &= \tfrac{1}{4}(g_{14} - g_{41} + g_{32} - g_{23}). \end{aligned} \tag{13}$$

Let the operator L_t correspond to the matrix $G_t(y)$ by formula (3). Then, by virtue of equality (4),

$$\operatorname{Re}\det\sigma(L_t) = \operatorname{Re}\det\sigma(L) + 4(1-t^2)\sum_{i=1}^{4}\left(\sum_{j=1}^{3}\lambda_{ij}\tau_j\right)^2 > 0, \tag{14}$$

where by $\lambda_{ij}(y)$ are denoted certain linear combinations of the elements of the matrix $G(y)$ of the type of formulas (13). In particular, for $t = 0$ formula (14) gives $g_1^2(y) + g_2^2(y) + g_3^2(y) + g_4^2(y) > 0$. Since $\varkappa(G)$ coincides with the index of the Noether operator L, $\varkappa(G)$ does not change under continuous (in Hölder' s sense) deformations of the matrix $G(y)$ preserving condition (5) (see (5), § 37,

§ 2 and (6)). Therefore $\varkappa(G) = \varkappa(G_0)$, and, without loss of generality, one may assume (see (6)) that

$$g_1^2(y) + g_2^2(y) + g_3^2(y) + g_4^2(y) = 1. \tag{15}$$

The boundary condition of the problem H_0' for a matrix of the form (1) satisfying condition (15) has the form

$$V^+(y) = \widetilde{G}(y)V^-(y), \tag{16}$$

where

$$\widetilde{G}(y) = \begin{pmatrix} g_1 & -g_2 & -g_3 & -g_4 \\ g_2 & g_1 & -g_4 & g_3 \\ g_3 & g_4 & g_1 & -g_2 \\ g_4 & -g_3 & g_2 & g_1 \end{pmatrix}.$$

Let $V(v_1, v_2, v_3, v_4)$ be a cogolomorphic vector satisfying condition (16). Then $U(-v_1, v_2, v_3, v_4)$ will be a solution of the problem H_0, and conversely. Consequently, $k = k'$ and $\varkappa(G_0) = 0$. Thus, we have proved

Theorem 2. *If condition* Γ *is fulfilled, the boundary-value problem* H *is Fredholm.*

Formulating the problem H for the upper and lower half-spaces $x_3 > 0$ and $x_3 < 0$ ($S = E_2$), one must additionally require that $U^-(y)$

and $f(y) \in L_p(E_2)$, $p > 1$. Theorems 1 and 2 remain valid* in this case as well.

Let us connect the Hilbert problem with the Riemann-Hilbert problem for the half-plane (problem Γ) (see ([6])). In ([6]) problem Γ is reduced to an equivalent system of singular integral equations

$$A\hat{\mu} + B\hat{P}\hat{\mu} = \hat{f}, \qquad \hat{\mu} = (\mu_1, \mu_2), \quad \hat{f} = (f_1, f_2), \tag{17}$$

a condition for the Noetherian property of this problem is indicated, and it is shown that, in computing the index of system (17), without loss of generality one may assume that

$$A(y) = \left\| \begin{matrix} g_1 & g_4 \\ -g_4 & g_1 \end{matrix} \right\|, \qquad B(y) = \left\| \begin{matrix} g_2 & g_3 \\ -g_3 & g_2 \end{matrix} \right\|,$$

where condition (15) is satisfied. The author takes this opportunity to correct an inaccuracy he made in ([6]) in proving the fact that $\chi(H_0) = 0$. The complete space E_2 is homeomorphic to the two-dimensional sphere S_2, so that one may assume that the functions g_1, g_2, g_3, g_4 realize a (continuous) mapping of the sphere S_2 into the sphere (15) S_3. However, every such mapping is unstable (see ([8])). Therefore, by an arbitrarily small change of the functions g_1, g_2, g_3, g_4 (the index of the corresponding Noether operator does not change), one can ensure that this mapping omits any point of S_3, for example the point $(-1, 0, 0, 0)$. Then the homotopy indicated in ([6]), which it is convenient to rewrite in the equivalent form $g_1^{(s)}(z) = \cos s\theta, \quad g_k^{(s)}(z) = g_k(z) \sin s\theta / \sin\theta \ (k = 2, 3, 4), \quad 0 \leq s \leq 1$, shows that system (17) is Fredholm.

The same assertion is not difficult to obtain from Theorem 2. Indeed, add to system (17) the equations $\mu_3 = f_3$ and $\mu_4 = f_4$, and denote by $\hat{H}\mu$ the operator generated by the left-hand side of the resulting system. Let $\hat{L}$ be the operator corresponding, by formula (3), to the matrix $\hat{G}_0$ of the form (1), where g_2 and g_3 are taken with the opposite sign. Then $R\hat{L}R^*\mu = \hat{H}\mu + K\mu$, where $K\mu$ is some completely continuous operator in the space $L_p(E_2)$, $p > 1$, while the simplest operator R is uniquely recovered from its symbolic matrix

$$\frac{1}{2}\left\|\begin{matrix} 1 & -i\tau_2 & i\tau_1 & 0 \\ 0 & i\tau_1 & i\tau_2 & 1 \\ -1 & -i\tau_2 & i\tau_1 & 0 \\ 0 & -i\tau_1 & -i\tau_2 & 1 \end{matrix}\right\| \qquad (\tau_1^2 + \tau_2^2 = 1, \quad i^2 = -1).$$

Since the operator R is invertible, $\chi(\hat{H}) = \chi(\hat{L})$ in the space $C_\alpha L_p(E_2)$, $p > 1$.

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* With the obvious modification of condition Γ.