ON AN APPROXIMATE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS FOR THE BIHARMONIC EQUATION

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.947.8

MATHEMATICS

A. V. BULEDZA

ON AN APPROXIMATE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS FOR THE BIHARMONIC EQUATION

(Presented by Academician A. A. Dorodnitsyn on 24 XII 1965)

For the boundary-value problem

$$\Delta\Delta u = f(x,y), \qquad u|_S = 0, \qquad \partial u/\partial n|_S = 0,$$

considered in the rectangle $\{0 \le x \le a,\ 0 \le y \le b\}$, we construct the difference analogue

$$(\Delta\Delta)_h u_{m,n} = f_{m,n}, \tag{1}$$

$$u_{m,n}|_{S_h} = 0, \qquad \partial u_{m,n}/\partial\nu|_{S_h} = 0, \tag{2}$$

$$m = 0,1,\dots,M; \quad n = 0,1,\dots,N; \quad h = a/M = b/N,$$

where $(\Delta\Delta)_h u_{m,n}$ is the well-known 13-point approximation of the biharmonic operator. If one-sided differences are used to represent $\partial u_{m,n}/\partial\nu|_{S_h}$, then conditions (2) will have the form

$$\begin{aligned} u_{m,0} = u_{m,1} = u_{m,N} = u_{m,N-1} = u_{0,n} = u_{1,n} = \\ = u_{M,n} = u_{M-1,n} = 0. \end{aligned} \tag{3}$$

Solving the eigenfunction and eigenvalue problem for the difference biharmonic operator, we obtain

$$u_{ks}(m,n) = Z(m,\varphi_k,v_k)Z(n,\psi_s,w_s), \tag{4}$$

$$Z(m,\varphi_k,v_k) = e^{v_k}\sin(m-1)\varphi_k - \sin m\varphi_k + e^{(1-m)v_k}\sin\varphi_k -$$

$$-e^{(m-M+2)v_k}\sin(M-2)\varphi_k + e^{(m-M+1)v_k}\sin(M-1)\varphi_k,$$

$$\lambda_{ks} = \frac{4}{h^4}\left[(2-\cos\varphi_k)^2 + (2-\cos\psi_s)^2 + 2\cos\frac{\pi k}{M-2}\cos\frac{\pi s}{N-2} - 4\right], \quad (5)$$

$$k = 1,2,\dots,M-3; \qquad s = 1,2,\dots,N-3,$$

where $\varphi_k, v_k, \psi_s, w_s$ are determined from the system

$$\sin(M-2)\varphi e^{2v} - 2\sin(M-1)\varphi e^{v} + \sin M\varphi = 0,$$

$$e^{2v} - 2(4-\cos\varphi)e^{v} + 1 = 0 \quad (6)$$

(to compute ψ_s, w_s, in (6) one should replace M by N, φ by ψ, and v by w).

It is easy to see that the eigenfunctions (4) asymptotically (with respect to M and N) satisfy the boundary conditions (3), i.e., conditions (3) are satisfied the more accurately the smaller the grid step. Moreover, the error decreases according to an exponential law.

It can also be shown that

$$(u_{ks}(m,n), u_{k's'}(m,n)) \to 0 \quad \text{as} \quad k \neq k',\ s \neq s',\ M \to \infty,\ N \to \infty, \quad (7)$$

where the convergence of the scalar product to zero occurs according to an exponential law.

The property defined by condition (7) will be called asymptotic orthogonality.

It is now easy to construct the difference Green' s function of problem (1), (2) (which exists by virtue of the positive definiteness of the difference operator)

$$G(m,n,\mu,\nu) = \sum_{k,s}\frac{u_{ks}(m,n)u_{ks}(\mu,\nu)}{\lambda_{ks}\|u_{ks}\|^2}$$

and to write the solution of this problem in the form

$$u(mh,nh) = \frac{h^4}{4}\sum_{\mu=2}^{M-2}\sum_{\nu=2}^{N-2}\left(\sum_{k,s}\frac{u_{ks}(m,n)u_{ks}(\mu,\nu)}{\lambda_{ks}\|u_{ks}\|^2}\right)f(\mu h,\nu h), \qquad (8)$$

where

$$\lambda_{ks} = \frac{h^4}{4}\Lambda_{ks}, \qquad \|u_{ks}\|^2 = (u_{ks},u_{ks}) = F(M,\varphi_k,\nu_k)F(N,\psi_s,w_s),$$

$$\begin{aligned}
F(M,\varphi,\nu) = {} & \frac{e^{2\nu}}{2}\left(M-2.5-\frac{\sin(2M-5)\varphi}{2\sin\varphi}\right) - e^{\nu}\left((M-3)\cos\varphi - \frac{\sin 2(M-2)\varphi - \sin 2\varphi}{2\sin\varphi}\right)\\
& + \frac{1}{2}\left(M-3-\frac{\cos M\varphi\,\sin(M-3)\varphi}{\sin\varphi}\right)\\
& + \frac{1}{e^{2\nu}-1}\left((e^{\nu}\sin(M-2)\varphi - \sin(M-1)\varphi)^2 + \sin^2\varphi\right)\\
& + \left(e^{2\nu}\sin^2(M-2)\varphi - e^{\nu}\sin(M-2)\varphi\sin(M-1)\varphi + \sin^2\varphi\right)\\
& + \frac{1}{e^{2\nu}-2e^{\nu}\cos\varphi+1}\Big(4e^{\nu}\sin\varphi\, 2\varphi - \sin^2 2\varphi + \frac{1}{2}\sin 4\varphi\\
& -2\sin^2\varphi\cos 2\varphi + 2(e^{2\nu}-1)(e^{\nu}\sin(M-2)\varphi\\
& -\sin(M-1)\varphi)(e^{\nu}\sin\varphi\cos(M-1)\varphi - (e^{\nu}\cos\varphi - 1)\sin(M-)\varphi)\Big)\,.
\end{aligned}$$

Let us note that in the expression for $\|u_{ks}\|^2$, small terms of the form $C_1^{(k)}e^{-M\nu_k}$, $C_2^{(k)}e^{-2M\nu_k}$, $C_1^{(s)}e^{-Nw_s}$, $C_2^{(s)}e^{-2Nw_s}$ have been discarded, where $C_i^{(k)}$, $C_i^{(s)}$, although they depend respectively on M and N, do not grow as M,N increase.

Formulas have also been obtained which make it possible to write the solution of the difference boundary-value problem in closed form (8) also in the case when the condition $\partial u/\partial n|_S$ is replaced by the condition $\Delta u|_S = 0$, or by all possible combinations of them on the sides of the rectangle.

Uzhgorod State
University

Received
24 XII 1965

Note: Figure translations are in progress. See original paper for figures.