# ASYMPTOTIC FORMULAS

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## Abstract

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**MATHEMATICS**

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# ASYMPTOTIC FORMULAS

# FOR A CERTAIN CLASS OF TRIGONOMETRIC SUMS

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In this paper an asymptotic formula is obtained for a certain class of trigonometric sums, making it possible to judge the behavior of the modulus of such sums as the interval of summation changes.

We shall use the following notation: $n, m, t, P, a$ are integers, $n \geqslant 20$; $r$ is a real number, $1 \leqslant r \leqslant 0.1n$; $t \geqslant n$; $p$ is a prime number, $(a, p) = 1$; $P \gg 1$;

$$\delta_n(x) = \begin{cases} 1, & \text{if } x \equiv 0 \pmod{n}, \\ 0, & \text{if } x \not\equiv 0 \pmod{n}. \end{cases}$$

**Theorem 1.** Let $q = p^t,\ P = q^{1/r}$. If

$$P = a_0 p^s + a_1 p^{s-1} + \cdots + a_{s-1} p + a_s, \quad 1 \leqslant a_0 \leqslant p-1, \quad 0 \leqslant a_\nu \leqslant p-1, \quad \nu \geqslant 1,$$

is the $p$-adic expansion of the number $P$, then the asymptotic formula

$$S = \sum_{x=1}^{P} \exp 2\pi i \, \frac{ax^n}{q} = A a_0 p^{s-\alpha} + a_0 p^{s-\alpha+\beta} + O\left(P^{1-\gamma/r^2}\right),$$

holds, where the quantities $A, \alpha, \beta$ are determined by the equalities

$$A = \delta_n(t-1) \sum_{x=1}^{p-1} \exp 2\pi i \, \frac{ax^n}{p}, \qquad \alpha = \left[\frac{t}{n}\right] + 1, \qquad \beta = \delta_n(t).$$

The formula has a particularly simple form when $P = p^s$ and $t \equiv 0 \pmod{n}$. In this case we obtain:

$$a_0 = 1, \quad a_\nu = 0, \quad \nu = 1, 2, \ldots, s; \qquad \delta_n(t-1) = 0, \quad A = 0, \quad \delta_n(t) = 1;$$

$$P^r = p^{sr} = p^t, \qquad s = t/r; \qquad t/ns = r/n;$$

$$S = P^{1-r/n} + O\Big(P^{1-\gamma/r^2}\Big).$$

For the proof of the theorem we shall need the following lemma.

**Lemma.** Let $q = p^m,\ m \geqslant n+1,\ (a_1, p) = \cdots = (a_{n+1}, p) = 1;$

$$P^r = q, \qquad 1 \leqslant r \leqslant 0.5n; \qquad f(x) = a_1 x + a_2 p x^2 + \cdots + a_{n+1} p^{n x^{n+1}}.$$

Consider the trigonometric sum

$$S_L = \sum_{x=1}^{P} \exp 2\pi i \, \frac{f(x)}{q}.$$

Then the estimate

$$|S_L| \leqslant C P^{1-\gamma_1/r^2},$$

holds, where $C$ and $\gamma_1 > 0$ are absolute constants.

The proof of this assertion is contained in (1), p. 239.

**Proof of Theorem 1.** Consider the sum

$$S' = \sum_{x=1}^{p^m} \exp 2\pi i \, \frac{a(x + bp^m)^n}{p^t}, \tag{1}$$

where $(a, p) = 1,\ r = t/m,\ 1 \leq r \leq 1/4n$, and obtain an asymptotic formula for it.

Let $t = t_1 n + t_2,\ 0 \leq t_2 \leq n-1$. We split the sum (1) into $t_1 - \delta_n(t) + 1$ sums, collecting together the terms in $x$ not divisible by $p$, divisible by $p$ but not by $p^2$, divisible by $p^2$ but not by $p^3$, and so on. According to this decomposition we obtain the equality

$$S' = p^{m - t_1 + \delta_n(t) - 1} + \sum_{\nu=0}^{t_1 - \delta_n(t)} S'_\nu,$$

where

$$S'_\nu = \sum_{\substack{x=1 \\ (x,p)=1}}^{p^{m-\nu}} \exp 2\pi i \, \frac{a(x + bp^{m-\nu})^n}{p^{t-\nu n}}, \qquad \nu = 0, 1, \dots, t_1 - \delta_n(t).$$

We split the sum over $\nu$ into two:

a) $0 \leq \nu \leq \nu'_0 = \left[\dfrac{t}{n-1}\left(1 - \dfrac{1}{r}\right)\right];$

b) $\nu'_0 + 1 \leq \nu \leq t_1 - \delta_n(t).$

Consider case a). In the sum $S'_\nu$ we make a linear change of the summation variable of the form

$$x = py + z, \qquad 1 \le z \le p - 1, \qquad 0 \le y \le p^{m-\nu-1} - 1.$$

Then we have the equality

$$x^n = (py + z + bp^{m-\nu})^n = a_0 + a_1 py + \dots + a_{np}^{ny^n,} \qquad (a_\nu, p) = 1,$$
$$\nu = 0, 1, 2, \dots, n.$$

For the modulus of the sum $S'_\nu$ we obtain the estimate:

$$|S'_\nu| \le p \left| \sum_{y=0}^{p^{m-\nu-1}-1} \exp 2\pi i \, \frac{a_1 y + a_2 p y^2 + \dots + a_{np}^{n-1} y^n}{p^{t-\nu n-1}} \right|.$$

Defining the quantity $r_\nu$ from the equality

$$(p^{m-\nu-1})^{r_\nu} = p^{t-\nu n-1},$$

we obtain

$$r_\nu = \frac{t - \nu n - 1}{m - \nu - 1} = r \, \frac{t - \nu n - 1}{t - \nu r - r}, \qquad 0 \le \nu \le \nu'_0.$$

It is easy to show that $1 \le r_\nu \le 2r, \ \nu = 0, 1, \dots, \nu'_0$. Applying the estimate of the lemma, after simple computations we arrive at the estimate

$$|S'_\nu| = O\left(P^{1-\gamma/r^2}\right).$$

Consider case b). We have

$$S'_\nu = \sum_{\substack{x=1 \\ (x,p)=1}}^{p^{m-\nu}} \exp 2\pi i \, \frac{a(x + bp^{m-\nu})^n}{p^{t-\nu n}}, \qquad \nu'_0 + 1 \le \nu \le t_1 - \delta_n(t).$$

For the selected values of $\nu$ the inequality holds:

$$m - \nu \ge t - \nu n, \qquad \nu = \nu'_0 + 1, \dots, t_1 - \delta_n(t).$$

Therefore, using the property of a complete trigonometric sum, we obtain

$$S'_\nu = p^{m-\nu-t+\nu n} \sum_{\substack{x=1\\(x,p)=1}}^{p^{t-\nu n}} \exp 2\pi i \,\frac{ax^n}{p^{t-\nu n}}, \qquad \nu'_0 + 1 \le \nu \le t_1 - \delta_n(t).$$

If $\delta_n(t) = 0$, then

$$l = t - \nu n \ge t - t_1 n = t_2 \ge 1.$$

If $t_2 \ge 2$, then $l \ge 2$ and

$$\sum_{\substack{x=1\\(x,p)=1}}^{p^l} \exp 2\pi i \,\frac{ax^n}{p^l} = 0.$$

If, however, $t_2 = 1$, then $S'_\nu = 0$ for all $\nu = \nu'_0 + 1, \dots, t_1 - 1$, and

$$S'_{t_1} = p^{m-t_1-1} \sum_{x=1}^{p-1} \exp 2\pi i \,\frac{ax^n}{p}.$$

If $\delta_n(t) = 1$, then $t_2 = 0$,

$$t = nt_1, \qquad \nu'_0 + 1 \le \nu \le t_1 - 1; \qquad t - \nu n \ge n \ge 2;$$

$$S'_\nu = 0, \qquad \nu = \nu'_0 + 1, \dots, t_1 - 1.$$

Consequently,

$$S' = p^{m-t_1-1} \delta_n(t-1) \sum_{x=1}^{p-1} \exp 2\pi i \,\frac{ax^n}{p} + p^{m-t_1-1+\delta_n(t)} + O\left(P^{1-\gamma/r^2}\right).$$

2. The equality is obvious

$$S = \sum_{\nu=0}^{s-1} \sum_{m=1}^{a_\nu p - a_\nu + 1} \left( \sum_{x=1}^{p^{s-\nu}-1} \exp 2\pi i \,\frac{a\left(x + b_m p^{s-\nu-1}\right)^n}{p^t} \right) + O(p),$$

where $b_m$ are certain integers.

For $\nu = 0, 1, \dots, \nu_0 = [s - 1 - 4t/n]$, for each of the inner sums there holds the asymptotic formula which we derived above. If $\nu > \nu_0$, then, estimating each of the remaining sums trivially, we obtain a remainder term of order $O(P^{1-\gamma/r^2})$. Thus the theorem is completely proved.

**Corollary.** For the sum $S$ the inequalities hold

$$S \ll \begin{cases} P^{1-\gamma r/n}, & \text{if } 1 \le r \le \sqrt[3]{n}, \\ P^{1-\gamma/r^2}, & \text{if } \sqrt[3]{n} \le r \le 0.1n. \end{cases}$$

The proof follows from Theorem 1 and simple calculations. Applying the estimates obtained, we get the assertion (see (2)).

**Theorem 2.** Consider the congruence

$$x_1^n + \ldots + x_k^n \equiv y_1^n + \ldots + y_k^n \pmod{p^t},$$

$$1 \le x_\nu, y_\nu \le P, \qquad \nu = 1, 2, \ldots, k.$$

For the number $I$ of solutions of this congruence, the following asymptotic formula holds for $k \ge c_1 n$:

$$I = \psi(p,k) P^{2k} p^{-t} + O\left(P^{2k-1} p^{-t}\right),$$

where

$$\psi(p,k) = \sum_{\nu=0}^{\infty} p^{-2k\nu} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^\nu} \left| \sum_{x=1}^{p^\nu} \exp 2\pi i \, \frac{ax^n}{p^\nu} \right|^{2k},$$

and the constant in $O$ depends only on $k$ and $t$.

This result cannot be improved, since it is easy to prove the inequality

$$I \gg P^{2k} p^{-t} p^{n-2k}.$$

Theorem 1 can be extended to a broader class of trigonometric sums.

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*Note: Figure translations are in progress. See original paper for figures.*

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