EXTREMAL QUASICONFORMAL MAPPINGS

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Abstract

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P. A. BILYUTA

EXTREMAL QUASICONFORMAL MAPPINGS

FOR ARBITRARY PLANE DOMAINS

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In the present paper, by means of variational methods for conformal (1) and quasiconformal (2) mappings, the following extremal problem is solved.

We shall consider q-quasiconformal mappings $w = f(z)$ of the disk $|z| < 1$ onto domains containing the points $0, a,\ 0 < a < 1$, with normalization $f(0) = 0,\ f(a) = a$. Let $F(w_1, w_2, \ldots, w_n)$ be a real function of the variables $w_k = f(z_k) = u_k + iv_k$, continuously differentiable with respect to u_k, v_k. It is required to find the mapping for which the function F assumes its maximum value for fixed values of $z_k,\ k = 1, 2, \ldots, n$.

By virtue of the normality of the family of mappings under consideration, the extremal mapping exists.

In what follows we shall need the following

Theorem 1 (G. M. Goluzin (1)). *If the function $w = f(z),\ f(0) = 0$, is regular and univalent in the disk $|z| < 1$, and the function $w^* = \Phi(z, \lambda)$, as a function of z and λ, is regular for $|\lambda| < \lambda_0$ and $r \le |z| < 1$, and for every $\lambda,\ 0 < \lambda < \lambda_0$, it is univalent in $r \le |z| < 1$; if, moreover, for every $z,\ r \le |z| < 1$, and small λ we have*

$$\Phi(z, \lambda) = f(z) + \lambda q(z) + O(\lambda^2),$$

then, adjoining to the image of the annulus $r < |z| < 1$ under the mapping by the function $w^* = \Phi(z, \lambda)$ the domain internal with respect to the image of the circle $|z| = r$, for small λ we obtain a simply connected domain D^*, containing the point $w^* = 0$, and for the function $w^* = f^*(z),\ f^*(0) = 0$, univalently mapping the disk $|z| < 1$ onto D^*, we have, for $|z| < 1$,

$$f^*(z) = f(z) + \lambda q(z) - \lambda z f'(z) S(z) + \lambda z f'(z) \overline{S(1/\bar{z})} + O(\lambda^2), \qquad (1)$$

where $S(z)$ is the sum of the terms with negative powers of z in the expansion of $q(z)/zf'(z)$ in the annulus $r < |z| < 1$.

The solution of the extremal problem stated above is given by

Theorem 2. *The function $w = f(z)$ for which $F(w_1, w_2, \dots, w_n)$ attains its maximum for fixed values z_k, $k = 1, 2, \dots, n$, has the following properties:*

1) *the mapping carried out by the function inverse to the extremal one has characteristics*

$$p(w) = q, \qquad \theta(w) = -\frac{1}{2}\arg A(w),$$

where

$$A(w) = \sum_{k=1}^{n} \frac{F_{w_k} w_k (a - w_k)}{\bar{w}(a - w)(w_k - w)}$$

(we leave aside the case $A \equiv 0$, corresponding to the presence of a stationary value of F: $F_{w_k} = 0$, $k = 1, 2, \dots, n$);

2) it maps the disk $|z| < 1$ onto the whole w-plane with cuts along a finite number of analytic arcs satisfying the inequality

$$A(w)\, dw^2 > 0. \tag{*}$$

1°. Let $w = f(z)$ be an extremal mapping. Subjecting the w-plane to a variation with constant characteristic h in the disk $K(\zeta, r) : |w - \zeta| \le r$, according to the formula

$$\omega = w + hr^2 \frac{w(a - w)}{\zeta(a - \zeta)(w - \zeta)} \quad \text{for } |w - \zeta| > r, \tag{2}$$

$$\omega = w + h(\overline{w} - \overline{\zeta}) \quad \text{for } |w - \zeta| \le r$$

(obviously, this variation does not take us out of the class of domains under consideration), we find that

$$\delta p = -2p|h| \cos 2(\theta^* - \theta), \tag{3}$$

$$dF = -2r^2 |hA| \cos 2(\theta^* - \theta_A), \tag{4}$$

where $\theta^* = \frac{1}{2}\arg h + \pi/2$, $\theta_A = -\frac{1}{2}\arg A(\zeta)$; p, θ are the characteristics of the mapping inverse to $w = f(z)$, at the point $w = \zeta$.

From (3) and (4) the first assertion of Theorem 2 follows immediately.

2°. From (4) we also conclude that the extremal domain D cannot have exterior points, since otherwise any variation (2) with constant characteristic h in a disk lying outside the domain would be admissible, and dF could have any sign.

Represent the extremal mapping $w = f(z)$, $f(0) = 0$, $f(a) = a$, in the form of a superposition of two mappings: 1) a quasiconformal mapping $\zeta = g(z)$ of the disk $|z| < 1$ onto the disk $|\zeta| < 1$, $g(0) = 0$, $g(a) = b$, where b, $0 < b < 1$, is a certain number, with characteristics $p(z) = q$, $\theta(z)$ of the mapping $w = f(z)$; and 2) a conformal mapping $w = \varphi(\zeta)$ of the disk $|\zeta| < 1$ onto the domain D, with $\varphi(0) = 0$, $\varphi(b) = a$.

First consider the function

$$w^* = w + hw/w_0(w - w_0), \tag{5}$$

where w_0 is an arbitrary finite point of the w-plane. For any prescribed $\rho > 0$ and sufficiently small complex h, the function (5) is univalent in the infinite domain $|w - w_0| > \rho$.

Now let the function $w = \varphi(\zeta)$, $\varphi(0) = 0$, $\varphi(b) = a$, be regular in the disk $|\zeta| < 1$ and map $|\zeta| < 1$ univalently onto the domain D. If $w_0 \in D$ and ζ_0 is from $|\zeta| < 1$ such that $w_0 = \varphi(\zeta_0)$, then for sufficiently small h the function

$$w^* = \varphi(\zeta) + h\varphi(\zeta)/\varphi(\zeta_0)(\varphi(\zeta) - \varphi(\zeta_0))$$

will be regular and univalent in some annulus $r \leq |\zeta| < 1$. For $\lambda = |h|$, Theorem 1 is applicable to this function. Since the function

$$\frac{q(\zeta)}{\zeta\varphi'(\zeta)} = e^{i\alpha}\frac{\varphi(\zeta)}{\zeta\varphi'(\zeta)\varphi(\zeta_0)(\varphi(\zeta) - \varphi(\zeta_0))}, \quad \alpha = \arg h,$$

in the disk $|\zeta| < 1$ has only a simple pole at the point ζ_0 with residue $e^{i\alpha}/\zeta_0\varphi'(\zeta_0)^2$, it is clear that

$$S(\zeta) = e^{i\alpha}/\zeta_0\varphi'(\zeta_0)^2(\zeta - \zeta_0),$$

and, consequently, formula (1) gives

$$\varphi^*(\zeta) = \varphi(\zeta) + h\frac{\varphi(\zeta)}{\varphi(\zeta_0)(\varphi(\zeta) - \varphi(\zeta_0))} - h\frac{\zeta\varphi'(\zeta)}{\zeta_0\varphi'(\zeta_0)^2(\zeta - \zeta_0)} +$$

$$+\overline{h}\frac{\zeta^2\varphi'(\zeta)}{\overline{\zeta_0\varphi'(\zeta_0)^2}(1 - \overline{\zeta_0}\zeta)} + O(|h|^2). \tag{6}$$

Let us further normalize the varied function (6), putting $\omega = a\varphi^*(\zeta)/\varphi^*(b)$, after which we obtain

$$\omega = \varphi(\zeta) + h\,\frac{\varphi(\zeta)(a-\varphi(\zeta))}{\varphi(\zeta_0)(a-\varphi(\zeta_0))(\varphi(\zeta)-\varphi(\zeta_0))} - h\,\frac{\zeta\varphi'(\zeta)}{\zeta_0\varphi'(\zeta_0)^2(\zeta-\zeta_0)} + h\,\frac{b\varphi'(b)\varphi(\zeta)}{a\zeta_0\varphi'(\zeta_0)^2(b-\zeta_0)} + \bar{h}\,\frac{\zeta^2\varphi'(\zeta)}{\zeta_0\varphi'(\zeta_0)^2(1-\bar{\zeta}_0\zeta)} - \bar{h}\,\frac{b^2\varphi'(b)\varphi(\zeta)}{a\zeta_0\varphi'(\zeta_0)^2(1-b\bar{\zeta}_0)} + O(|h|^2). \tag{7}$$

This is the variational formula for functions $w = \varphi(\zeta)$ mapping conformally the disk $|\zeta| < 1$ onto the domain D with normalization $\varphi(0) = 0,\ \varphi(b) = a$.

If now $w = \varphi \circ g$ is an extremal mapping, then, varying the function $\varphi(\zeta)$ according to formula (7), we obtain

$$dF = 2\,\mathrm{Re}\,h \sum_{k=1}^{n} \left[\frac{F_{w_k} w_k (a-w_k)}{\varphi(\zeta_0)(a-\varphi(\zeta_0))(w_k-\varphi(\zeta_0))} - \frac{F_{w_k}\zeta_k w_k'}{\zeta_0\varphi'(\zeta_0)^2(\zeta_k-\zeta_0)} + \frac{F_{w_k} w_k b\varphi'(b)}{a\zeta_0\varphi'(\zeta_0)^2(b-\zeta_0)} + \frac{\overline{F}_{w_k}\bar{\zeta}_k^2\bar{w}_k'}{\zeta_0\varphi'(\zeta_0)^2(1-\zeta_0\bar{\zeta}_k)} + \frac{\overline{F}_{w_k}\bar{w}_k b^2\varphi'(b)}{a\zeta_0\varphi'(\zeta_0)^2(1-b\bar{\zeta}_0)} \right] \le 0,$$

where

$$\zeta_k = g(z_k), \qquad w_k = \varphi(\zeta_k), \qquad w_k' = \varphi'(\zeta_k).$$

By virtue of the arbitrariness of $\arg h$ and ζ_0, we conclude that in the case of an extremal mapping the function $\varphi(\zeta)$ satisfies the differential equation

$$\frac{\zeta\varphi'(\zeta)^2}{\varphi(\zeta)(a-\varphi(\zeta))} \sum_{k=1}^{n} \frac{F_{w_k} w_k (a-w_k)}{1-\bar{\zeta}\zeta_k} =$$

$$= \sum_{k=1}^{n} \left[\frac{F_{w_k}\zeta_k w_k'}{\zeta_k-\zeta} - \frac{F_{w_k} w_k b\varphi'(b)}{a(b-\zeta)} - \frac{\overline{F}_{w_k}\bar{\zeta}_k^2\bar{w}_k'}{1-\zeta\bar{\zeta}_k} + \frac{\overline{F}_{w_k}\bar{w}_k b^2\varphi'(b)}{a(1-b\zeta)} \right].$$

From the analytic theory of differential equations it follows that $\varphi(\zeta)$ is regular not only in the disk $|\zeta| < 1$, but also on the circumference $|\zeta| = 1$, except for a finite number of points. Consequently, the boundary of the domain D consists of a finite number of analytic arcs.

3°. Let us prove that the slits forming the boundary of the extremal domain D satisfy inequality $(*)$.

Suppose, to the contrary, that at some point of a slit $w = w_0$, distinct from its end, the angle between the direction $dw_0 = -\frac{1}{2}\arg A(w_0)$ and the slit is equal to $\theta,\ 0 < \theta < \pi$. In view of the analyticity of the boundary and of the fact that we shall vary the w-plane in a sufficiently small neighborhood of the point w_0, one may regard the arc of the slit lying in this neighborhood as a rectilinear segment; one may also take $w_0 = 0$.

We now vary the w-plane as follows. Outside the segment

$$u^2 + (v + c\operatorname{ctg}\alpha)^2 < c^2/\sin^2\alpha, \qquad v > 0$$

the mapping is identical, and we contract the segment $1/k$ times in the direction θ; here $c > 0$ is sufficiently small, while $\alpha,\ 0 < \alpha < \pi/2$, and $k,\ 0 < k < 1$, will be chosen later. This mapping $\omega = \omega(w)$, $\omega = \xi + i\eta$, has the form

$$\eta = (k\sin^2\theta + \cos^2\theta)v - (1-k)\sin\theta\,[u\cos\theta + c\operatorname{ctg}\alpha\sin\theta -$$

$$-\sqrt{c^2(1+\operatorname{ctg}^2\alpha) - (v\cos\theta - u\sin\theta + c\operatorname{ctg}\alpha\cos\theta)^2}],$$

$$\xi = (\eta - v)\operatorname{ctg}\theta + u.$$

The condition that, under the additional mapping $\omega = \omega(w)$, the characteristic p decreases is as follows:

$$(\xi_u^2 + \eta_u^2 - \xi_v^2 - \eta_v^2)\cos 2\theta + 2(\xi_u\xi_v + \eta_u\eta_v)\sin 2\theta < 0. \tag{8}$$

For small α we have

$$\xi_u = 1 + O(\alpha), \qquad \xi_v = -(1-k)\operatorname{ctg}\theta + O(\alpha),$$

$$\eta_u = O(\alpha), \qquad \eta_v = k + O(\alpha),$$

and condition (8) is written as

$$-(1-k)\,[2 + (1-k)\cos 2\theta/\sin^2\theta] + O(\alpha) < 0.$$

For any $\theta,\ 0 < \theta < \pi$, this inequality can be satisfied if k is taken so close to 1 that the expression in square brackets is positive, and α is sufficiently small.

Thus, we have constructed a mapping that has not changed the value of F, but maps onto a domain with exterior points. This contradiction proves our assertion, which, together with the results of item 2°, completes the proof of Theorem 2.

Remark. Obviously, the same extremal problem can be considered for q-quasiconformal mappings $w = f(z)$ of an arbitrary simply connected domain D of the z-plane with the normalization $f(z') = 0,\ f(z'') = a$, where $z', z'' \in D$, and in this case Theorem 2 holds for the extremal function.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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CITED LITERATURE

1. G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow, 1952.

2. P. P. Belinskii, DAN, 121, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.