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ON THE SOLUTION OF A ONE-PHASE PROBLEM OF STEFAN TYPE

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Abstract

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MATHEMATICAL PHYSICS

L. I. RUBINSHTEIN

ON THE SOLUTION OF A ONE-PHASE PROBLEM OF STEFAN TYPE

(Presented by Academician A. A. Dorodnitsyn on 20 IX 1965)

Below we prove the existence of a solution of the problem

$$a^2(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < y(t), \quad x \neq 1, \quad t > 0; \tag{1_1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad u|_{t=0} = 1, \quad 0 < x < 1, \qquad u|_{x=y(t)} = 0; \tag{1_2}$$

$$u \text{ and } k\frac{\partial u}{\partial x} \quad \text{are continuous for } x = 1, \quad t > 0; \tag{1_3}$$

$$\dot{y}(t) = -\alpha + \beta \left.\frac{\partial u}{\partial x}\right|_{x=y(t)}, \qquad t > 0, \qquad y(0) = 1; \tag{1_4}$$

$$a(x) = \begin{cases} a_1 = 1, & 0 < x < 1, \\ a_2 = \text{const} > 0, & x > 1; \end{cases}$$

$$k(x) = \begin{cases} k_1 = \text{const} > 0, & 0 < x < 1, \\ k_2 = \text{const} > 0, & x > 1, \end{cases} \qquad \alpha = \text{const} > 0, \quad \beta = \text{const} < 0, \tag{1_5}$$

such that

$$v(t) \equiv \frac{\partial}{\partial x} u(y(t), t) = \frac{w(t)}{\sqrt{t}} = \frac{w(0)}{\sqrt{t}} + v_1(t), \tag{1_6}$$

where $v_1(t)$ is continuous for $t \geq 0$. The uniqueness of its solution was shown in ([1]). There, too, a physical interpretation of this problem was indicated.

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The existence of a solution of problem (1_i) $(i = 1, \dots, 6)$ follows from the following lemmas.

Lemma 1. Let $\gamma > 0$ be a root of equation (35) from ([1]); let γ_1 and γ_2 be arbitrary positive numbers such that $\gamma_1 < \gamma < \gamma_2$. Further, let $U_j(x,t)$ be defined by the conditions (1_i) $(i = 1, 2, 3)$, if in them $y(t)$ is replaced by $z_j(t) = 1 + 2\alpha\gamma_j\sqrt{t}$ $(j = 1, 2)$. Then there exists $T > 0$ such that

$$\dot{z}_1(t) < \frac{\partial U_1}{\partial x}\bigg|_{x=z_1(t)}, \qquad \dot{z}_2(t) > \frac{\partial U_2}{\partial x}\bigg|_{x=z_2(t)}, \qquad t \in (0, T]. \tag{2}$$

Lemma 2. Let $T > 0$ and let $z_j(t)$ be defined according to Lemma 1. Consider a sequence $\{t_n\}$, monotonically decreasing to zero, $t_0 \le T$. Then for any $n \ge 0$ and for all $t \in (t_n, T)$ there exists a unique solution $u_n(x,t)$, $y_n(t)$ of the problem obtained from problem (1_i) by replacing condition (1_3) by the condition

$$\dot{y}(t) = -\alpha + \beta \frac{\partial u}{\partial x}\bigg|_{x=y(t)} \quad \text{for } t_n < t,$$

$$y(t) \equiv z_1(t) \quad \text{for } 0 \le t \le t_n. \tag{3}$$

Here $y_n(t)$ is differentiable for $t > t_n$.

Lemma 3. Let u_n, y_n be defined according to Lemma 2. Then there exists $T_0 \in (0, T]$ such that, for any $n \ge 0$ and for $0 < t \le T_0$,

$$\dot{y}_n(t) > 0, \qquad -\frac{1}{z_1(t) - 1} \le -\frac{1}{y_n(t) - 1} < \frac{\partial u_n}{\partial x}\bigg|_{x=y_n(t)} < 0. \tag{4}$$

Suppose that Lemmas 1-3 have been proved, and let us prove the existence of a solution of the original problem. First of all, note that from the uniqueness of u_n, y_n satisfying the conditions of Lemma 2 it follows that $u_n \equiv U_1$ for $0 < x < z_1(t)$, $0 \le t \le t_n$. Hence, and from Lemma 1, it follows that

$$\dot{y}_n(t_n) > \dot{z}_1(t_n), \qquad y_n(t_n) = z_1(t_n). \tag{5}$$

But this means that on some interval $(t_n, t_n + \delta)$ one certainly has

$$z_1(t) < y_n(t) < z_2(t). \tag{6}$$

At the same time, from the definition of $y_n(t)$ it follows that there exists an interval $(t_{n+1}, t_{n+1} + \delta^*)$ on which $y_{n+1}(t) > y_n(t)$. Assuming further that the set $\mathcal{E}_n = \{t \in (0, T),\, y_{n+1}(t) < y_n(t)\}$ is nonempty, and putting $\tau_n = \inf \mathcal{E}_n$, we find, using the maximum principle and Friedman' s theorem (2), that

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$\dot{y}_{n+1}(\tau_n) - \dot{y}_n(\tau_n) > 0$. But this contradicts the definition of τ_n; consequently, $\mathcal{E}_n$ is empty, i.e.

$$y_{n+1}(t) \geq y_n(t) \qquad \text{for } 0 < t \leq T, \quad n = 0, 1, 2, \dots . \tag{7}$$

In exactly the same way we prove that for all n and $t \in (0, T)$

$$z_1(t) \leq y_n(t) \leq z_2(t). \tag{8}$$

Note that this conclusion does not rely on Lemma 3.

Thus, $\{y_n(t)\}$ is monotone and bounded. Moreover, it is uniformly continuous, since from Lemma 3 it follows that $\{\sqrt{t}\dot{y}_n(t)\}$ is uniformly bounded. Consequently, $\{y_n(t)\}$ converges to a function $y^*(t)$ satisfying the Lipschitz condition

$$|y^*(t) - y^*(\tau)| < A\,\frac{|t-\tau|}{\sqrt{t}+\sqrt{\tau}}, \qquad 0 < t, \tau < T. \tag{9}$$

From the monotone convergence of $\{y_n\}$ to y^* and from the maximum principle it follows that $\{u_n(x,t)\}$ converges to a bounded solution $u^*(x,t)$ of problem (1_i) $(i = 1, 2, 3)$, in which the boundary coincides with $y^*(t)$. Moreover, from (9) follows the existence of $v^*(t) = \dfrac{\partial}{\partial x} u^*(y^*(t), t)$, and $v^*(t)$ is continuous for $t > 0$.

It is easy to verify that

$$y^*(t) = 1 - \alpha t + \beta \int_0^t v^*(\tau)\, d\tau, \tag{10}$$

i.e., that u^*, y^* satisfy all the conditions of problem (1_i) $(i = 1, \dots, 6)$. Indeed, equation (1_1) entails the integral identity

$$\int_0^t d\tau \int_0^{y(\tau)} k(x) \frac{\partial^2 u^*}{\partial x^2}\, dx = \int_0^t d\tau \int_0^{y^*(\tau)} \frac{k(x)}{a^2(x)} \frac{\partial u^*}{\partial \tau}\, dx. \tag{11}$$

Carrying out the integration and taking (12) into account, we find that

$$k_2 \int_0^t v^*(\tau)\, d\tau = k_1 \int_0^1 (u^*(x,t) - 1)\, dx + \frac{k_2}{a_2^2} \int_1^{y^*(t)} u^*(x,t)\, dx. \tag{12}$$

On the other hand, from the conditions of Lemma 2 it similarly follows that

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$$k_2 \int_0^t \frac{\partial u_n}{\partial x}\bigg|_{x=y_n(t)} d\tau = k_1 \int_0^1 (u_n(x,t)-1)\,dx + \frac{k_2}{a_2^2}\int_1^{y_n(t)} u_n(x,t)\,dx. \quad (13)$$

Using condition (3), we rewrite (13) in the form

$$k_2 \int_0^{t_n} \frac{\partial U_1}{\partial x}\bigg|_{x=y_n(\tau)} d\tau + \frac{k_2}{\beta}(y_n(t)+\alpha t) - \frac{k_2}{\beta}\left[z_1(t_n)+\alpha t_n\right]$$
$$= k_1 \int_0^1 (u_n(x,t)-1)\,dx + \frac{k_2}{a_2}\int_1^{y_n(t)} u_n(x,t)\,dx. \quad (14)$$

Passing here to the limit as $n \to \infty$ and using the uniform convergence of u_n and y_n to u^* and y^*, respectively, we obtain

$$\frac{k_2}{\beta}(y^*(t)+\alpha t) = k_1 \int_0^1 (u^*(x,t)-1)\,dx + \frac{k_2}{a_2^2}\int_1^{y^*(t)} u^*(x,t)\,dx. \quad (15)$$

Comparing (15) and (11), we arrive at (10), as was required to prove. Thus, in order to prove the theorem on the existence of a solution, it remains to prove the validity of Lemmas 1-3.

The validity of Lemma 1 follows from the following considerations. If

$$W_i(t) = \sqrt{t}\,\frac{\partial U_i}{\partial x}\bigg|_{x=z_i},$$

then, as in ([1]), it is easy to prove that

$$W_i(0) = -\frac{\exp(-\gamma_i^2)}{(\lambda+\operatorname{Erf}\gamma_i)a\sqrt{\pi}} \equiv f(\gamma_i)\cdot\left(\lambda = \frac{k_2}{a_2^2 k_1}\right). \quad (16)$$

But $f(\gamma) < 0$ for $\gamma > 0$. Hence, also from (35) in ([1]), it follows that

$$\beta W_2(0) < \gamma a < \beta W_1(0).$$

But this means that

$$\lim_{t\to 0}\sqrt{t}\,\beta\,\frac{\partial U_2}{\partial x}\bigg|_{x=z_2} < \lim_{t\to 0}\sqrt{t}\,\dot{z}_2(t), \qquad \lim_{t\to 0}\sqrt{t}\,\beta\,\frac{\partial U_1}{\partial x}\bigg|_{x=z_1} > \lim_{t\to 0}\sqrt{t}\,\dot{z}_1(t), \quad (17)$$

whence the assertion of the lemma follows.

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We prove the validity of Lemma 3 under the assumption that Lemma 2 is valid. Applying to u_n the strong maximum principle and Friedman' s theorem ([2]), we find that $0 \le u_n \le 1$ for $0 \le t \le T$. Suppose that on the interval $0 < t < T_n$ one has $\dot{y}_n > 0$, and consider the function

$$\bar{u}_n(x,t) = 1 - \frac{x-1}{y_n(t)-1}, \qquad 1 < x < y_n(t), \qquad 0 < t \le T_n. \tag{18}$$

In the indicated domain $\bar{u}_n$ is superparabolic, vanishes for $x = y_n(t)$, and is equal to 1 for $x = 1$. Consequently, by the maximum principle, $\bar{u}_n \ge u_n$ for $1 < x < y_n(t)$, $0 < t \le T_n$, and the equality sign is certainly attained for $x = y_n(t)$. But this means that

$$-\frac{1}{y_n(t)-1} < \left.\frac{\partial u_n}{\partial x}\right|_{x=y_n(t)} < 0. \tag{19}$$

Using now the inequality (8), proved without reference to Lemma 3, we shall verify the validity of the second assertion in (3), provided only that

$$\tau_n^* = \inf_n T_n > 0. \tag{20}$$

To prove the validity of (20), note that the reduction of the problem of determining $u_n(x,t)$ to an integral equation, similar to equation (33) of ([1]), allows one to show that

$$\lim_{t\to 0} U_1(1,t) > 0. \tag{21}$$

Hence, from the following consequence of the maximum principle and Friedman' s theorem, namely the inequality $u_n \ge U_1$ $(n = 0, 1, 2, ...)$, there follows the existence of $m > 0$

such that

$$u_n|_{x=1} \ge m > 0 \quad \text{for } 0 \le t \le T_1^* \le T. \tag{22}$$

Here T_1^* does not depend on the choice of n. Let us now take $k > -\beta/a_2^2$ and consider the function

$$\underline{u}_n(x,t) = m\exp(-k)\left[\exp\left(k - k\,\frac{x-1}{y_n-1}\right) - 1\right]. \tag{23}$$

It is easy to see that for $1 < x < y_n(t)$, $0 < t \le T_n$, $\underline{u}_n$ is subparabolic. Further,

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$$\underline{u}_n\big|_{x=0} = m\exp(-k)[\exp k - 1] < m, \qquad \underline{u}_n\big|_{x=y_n(t)} = 0.$$

Consequently, for $0 \le t \le T_1^*$ the boundary values of $\underline{u}_n$ are less than the boundary values of u_n. Hence, from the uniform boundedness of $\underline{u}_n$ and u_n, the parabolicity of u_n, the subparabolicity of $\underline{u}_n$, and the maximum principle, it follows that

$$\underline{u}_n(x,t) \le u_n(x,t) \quad \text{for } 1 < x < y_n(t), \quad 0 < t \le \min(T_1^*, T_n). \tag{24}$$

Since, moreover, $u_n = \underline{u}_n = 0$ when $x = y_n(t)$, $t > 0$, it follows from (24) that

$$\frac{\partial}{\partial x} u_n\Big|_{x=y_n(t)} \le -\frac{mk}{y_n(t) - 1} \quad \text{for } 0 < t \le \min(T_1^*, T_n). \tag{25}$$

Majorizing (25) by means of (8), we find

$$\frac{\partial u_n}{\partial x}\Big|_{x=y_n(t)} \le -\frac{mk}{z_2(t) - 1} \quad \text{for } 0 < t \le \min(T_1^*, T_n). \tag{26}$$

Finally, substituting (26) into the first of conditions (3), we obtain

$$\dot{y}_n(t) > -\alpha - \frac{mk\beta}{z_2(t) - 1} \equiv f(t) \quad \text{for } 0 < t \le \min(T_1^*, T_n). \tag{27}$$

Let now T_2^* be determined from the condition $f(t) > 0$ for $0 \le t \le T_2^*$. Since $z_2(0) = 1$, we see that $T_2 > 0$. But T_2^* and T_1^* are defined independently of n, and, obviously, $T_n \ge T_2^*$. Therefore, T_n may be defined by the equality $T_n = \min(T_1^*, T_2^*)$, and this means that $\tau_n \ge \min(T_1, T_2) > 0$. The lemma is proved.

Thus, it remains to prove Lemma 2. But its proof is an obvious repetition of the analysis of the integral equations of the classical Stefan problem, carried out by us earlier (see ([3],[4])). One need only, as follows from (1), use the Green' s function for the two-layer problem on the half-line. Therefore we shall not dwell on it.

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Note: Figure translations are in progress. See original paper for figures.

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