

# DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DERIVATIVES AND STOCHASTIC EQUATIONS FOR GENERALIZED RANDOM PROCESSES

MATHEMATICS

1966

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.80966>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

## Abstract

## Full Text

UDC 513.88+517.948+519.217

MATHEMATICS

Yu. L. DALETSKII

# DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DERIVATIVES AND STOCHASTIC EQUATIONS FOR GENERALIZED RANDOM PROCESSES

(Presented by Academician A. Yu. Ishlinskii, June 3, 1965)

1°. Let  $\mathfrak{h}$  be a Hilbert space, and let  $T$  be a positive definite operator with dense domain of definition  $D$  in  $\mathfrak{h}$ , such that the operator  $T^{-1}$  is a Hilbert–Schmidt operator.

We transform  $D$  into a Hilbert space  $\mathfrak{h}_+$ , introducing in it the positive norm  $\|\varphi\|_+ = \|T\varphi\|$ . Let  $\mathfrak{h}_- \supset \mathfrak{h}$  be the corresponding space with negative norm, defined by the formula  $\|\xi\|_- = \|T^{-1}\xi\|$  for  $\xi \in \mathfrak{h}$ , obtained from  $\mathfrak{h}$  by completion in this norm. (The operator  $T^{-1}$  is extended to all of  $\mathfrak{h}_-$  so that the last formula remains valid also for  $\xi \in \mathfrak{h}_-$  (for details see <sup>(1–3)</sup>.)

We shall consider nonlinear functionals  $F(x)$  in  $\mathfrak{h}$ , admitting an extension to  $\mathfrak{h}_-$  and twice differentiable there in the sense that the representation

$$F(x+h) = F(x) + F_1(x;h) + \frac{1}{2}F_2(x;h,h) + o(\|h\|_-^2)$$

is valid for  $x, h \in \mathfrak{h}_-$ , where  $F_1(x;h)$  is a linear, and  $F_2(x;h_1, h_2)$  a bilinear, functional in  $\mathfrak{h}_-$  for each  $x \in \mathfrak{h}_-$ . It is easy to see that  $F_1(x;h) = (F'(x), h)$ ,  $F_2(x;h_1, h_2) = (F''(x)h_1, h_2)$ , where  $h, h_1, h_2 \in \mathfrak{h}_-$ ;  $F'(x) \in \mathfrak{h}_+$ ;  $F''(x)$  is a linear operator acting from  $\mathfrak{h}_-$  to  $\mathfrak{h}_+$  and having finite trace. We shall call  $F'(x)$  and  $F''(x)$ , respectively, the first and second derivatives of the functional  $F(x)$ . Functionals for which these derivatives are continuous at each point  $x \in \mathfrak{h}_-$  in the corresponding norms will be called functionals of class  $C_2$ . For functionals of this class we introduce differential operators of the first and second orders by the formulas

$$L_1(F) = (F'(x), a(x)), \quad L_2(F) = \text{Sp}[B(x)F''(x)],$$

where  $a(x) \in \mathfrak{h}_-$ ,  $B(x)$  is a bounded linear operator in  $\mathfrak{h}$ .

The operator  $L_2(F)$  is naturally regarded as elliptic when the operator  $\text{Re } B(x)$  is sign-definite, and as the Laplace operator when  $B(x) = I$ .

2°. The purpose of this article is to consider equations of diffusion type

$$\partial F / \partial t = \text{Sp}[B(x, t)F''] + (F', a(x, t)) + v(x, t)F. \quad (1)$$

Here the operator  $B(x, t)$  is assumed positive definite.

We first consider the simpler equation

$$\partial F / \partial t = \text{Sp}[BF''] \quad (2)$$

with constant operator  $B$ .

Let  $P$  be the operator of orthogonal projection onto a finite-dimensional subspace  $\mathcal{L} \subset \mathfrak{h}_+$ . We introduce on the cylindrical sets of the space  $\mathfrak{h}_-$  with base in  $\mathcal{L}$  the Gaussian measure:

$$\mu_P(x, t; S) = [(2\pi t)^n \det(PBP)]^{-1/2} \int_{PS} \exp \left\{ -\frac{1}{2t} ([PBP]^{-1}(y-x), (y-x)) \right\} dy.$$

By virtue of a known result of R. A. Minlos (see <sup>(4)</sup>), in the space  $\mathfrak{H}_-$ , on the  $\sigma$ -ring  $\mathfrak{R}$  containing all cylindrical sets, there exists a Gaussian measure  $\mu(x, t; S)$  with mean value  $x$  and correlation operator  $Bt$ , which is an extension of the measures  $\mu_P$ .

**Theorem 1.** Let  $\Phi(x)$  be a continuous functional on  $\mathfrak{H}_-$  satisfying the condition

$$|\Phi(x)| \leq C_\alpha e^{\alpha(B^{-1}x, x)} \quad (C_\alpha > 0, \alpha > 0). \quad (3)$$

There exists an interval  $(0, t)$  on which the Cauchy problem for equation (2) with initial condition  $F(x, 0) = \Phi(x)$  has a solution representable in the form

$$F(x, t) = \int_{\mathfrak{H}_-} \Phi(y) \mu(x, t; dy). \quad (4)$$

This solution is unique in the class of functionals from  $C_2^-$  which, together with the derivative  $\partial F / \partial t$ , satisfy condition (3). If condition (3) is fulfilled for every  $\alpha > 0$ , then these assertions are valid for arbitrarily large  $T$ .

3°. Let  $C(\mathfrak{H}_-)$  be the space of continuous functions  $\xi(t)$  ( $0 \leq t \leq \tau$ ) with values in  $\mathfrak{H}_-$ . By  $C_{x, \tau}(\mathfrak{H}_-)$  we denote its subspace consisting of functions satisfying the condition  $\xi(\tau) = x$ . Let  $q$  ( $0 < \tau_1 < \dots < \tau_s < \tau$ ) be a partition of the interval  $(0, \tau)$ , and let  $M_q$  be a cylindrical set in  $C_{x, \tau}(\mathfrak{H}_-)$ , defined by the partition  $q$  and by a collection of sets  $M_j \subset \mathfrak{H}_-$  ( $j = 1, \dots, s$ ):

$$M_q = \{\xi(t) : \xi(\tau_j) \in M_j, j = 1, 2, \dots, s\}.$$

On the ring of such cylindrical sets introduce the measure

$$m(M_q) = \int_{\mathfrak{H}_- M_q} \Phi(y) \mu(y_1, \tau_1, dy) \mu(y_2, \tau_2 - \tau_1, dy_1) \cdots \mu(x, \tau - \tau_s, dy_s),$$

where  $\Phi(y)$  is a functional possessing the same properties as in Theorem 1. Using the results from (5), one can show that this measure can be extended to some  $\sigma$ -ring in  $C_{x,\tau}(\mathfrak{H}_-)$  containing all cylindrical sets.

**Theorem 2.** Let  $V(x, t)$  be a bounded, continuous functional on  $\mathfrak{H}_-$ . The solution of the Cauchy problem

$$\partial F / \partial t = \text{Sp}[BF''] + V(x, t)F, \quad F(x, 0) = \Phi(x),$$

is representable in the form of the integral

$$F(x, t) = \int_{C_{x,\tau}(\mathfrak{H}_-)} \exp \left\{ \int_0^\tau v(\xi(u), u) du \right\} dm.$$

4°. To construct a solution of equation (1) one can apply the theory of stochastic equations developed by I. I. Gikhman, K. Ito, and A. V. Skorokhod (see (6)). V. V. Baklan (7) and T. L. Chantladze (8) considered stochastic equations in Hilbert space and, by this method, constructed the solution of equations in functional derivatives of a certain special form, corresponding to the case when the operator  $B$  in (1) has finite trace. Below we make essential use of the results from (7, 8) in order to investigate a more general stochastic equation in a space with negative norm. The solution of such an equation is, generally speaking, a generalized random process.

Thus, we shall consider random variables with values in the space  $\mathfrak{H}_-$ . In the case when  $\mathfrak{H}$  is a space of functions, they are generalized random functions (processes or fields).

Let  $w(t)$  be a homogeneous random process with independent increments, taking values in  $\mathfrak{H}_-$  and possessing the following property: for any collection  $\varphi_j \in \mathfrak{H}_+$  ( $j = 1, \dots, n$ ), the  $n$ -dimensional random

quantity  $(w(t_1) - w(t_2), \varphi_j)$  is a Gaussian random variable with zero mean and correlation matrix  $(t_1 - t_2) \|(\varphi_k, \varphi_j)\|_{j,k=1}$ .

Under the usual assumptions, stochastic integrals of the form

$$\int_0^\tau A(t) dw(t), \quad \int_0^\tau (\alpha(t), dw(t)),$$

are introduced in the usual way, where  $A(t), \alpha(t)$  are random functions whose values are, respectively, operators in  $\mathfrak{H}_-$  and vectors in  $\mathfrak{H}_-$ . These integrals obey the estimates

$$\begin{aligned} \mathbf{M} \left\| \int_0^\tau A(t) dw(t) \right\|_-^2 &\leq (\text{Sp } T^{-2}) \cdot \mathbf{M} \left\| \int_0^\tau (T^{-1}AT)^*(T^{-1}AT) dt \right\| \leq \\ &\leq (\text{Sp } T^{-2}) \int_0^\tau \mathbf{M} \|A(t)\|^2 dt, \\ \mathbf{M} \left| \int_0^\tau (\alpha(t), dw(t)) \right|^2 &\leq \int_0^\tau \mathbf{M} \|\alpha(t)\|^2 dt. \end{aligned}$$

Consider the stochastic equation

$$\xi(u) = \xi(t) + \int_t^u a(\xi(s), s) ds + \int_t^u A(\xi(s), s) dw(s) \quad (0 \leq t \leq u \leq \tau), \quad (5)$$

where  $\xi(s) \in \mathfrak{H}_-$ , and  $a(\xi, s)$  and  $A(\xi, s)$  are measurable functions on  $\mathfrak{H}_- \times [0, \tau]$ , whose values are, respectively, vectors from  $\mathfrak{H}_-$  and bounded operators acting in  $\mathfrak{H}_-$ .

**Theorem 3.** *Suppose that the functions  $a(\xi, s)$  and  $A(\xi, s)$  satisfy in  $\mathfrak{H}_-$  a Lipschitz condition with a constant independent of  $\xi$  and  $s$ .*

*If  $\mathbf{M} \|\xi(t)\|_-^2 < \infty$ , then there exists a unique, up to stochastic equivalence, solution of equation (5) satisfying the condition*

$$\int_t^\tau \|\xi(u)\|_-^2 du < \infty.$$

*This solution is a Markov process with values in  $\mathfrak{H}_-$ .*

The proof of the theorem is based on the expansion, convergent in  $\mathfrak{H}_-$ ,

$$w(t) = \sum_{k=1}^{\infty} w_k(t) \varphi_k$$

in the eigenvectors  $\varphi_k$  of the operator  $T$ . Here

$$w_k(t) = (\varphi_k, w(t)) \quad (k = 1, 2, \dots)$$

is a sequence of mutually independent Wiener processes.

Under some additional assumptions on the smoothness of the coefficients  $a$  and  $A$ , one can show that the backward Kolmogorov equation for the process  $\xi(t)$  satisfying equation (5) is an equation of type (1),

$$\partial F / \partial t + \text{Sp}(A^* F'' A) + (F', a) = 0. \quad (6)$$

In this case the solution of the Cauchy problem for equation (5) is expressed in terms of integrals with respect to the measure determined by the random process  $\xi(t)$ .

**Theorem 4.** *Suppose that the coefficients  $a(\xi, t)$ ,  $A(\xi, t)$ , and the functional  $\Phi(\xi)$  have, in  $\mathfrak{H}_-$ , derivatives with respect to  $\xi$  up to and including second order, satisfying the Lipschitz condition, and let  $\mu(x, t, u, S)$  be the measure generated in  $\mathfrak{H}_-$  by the solution of equation (5) with  $\xi(t) = x$ .*

Then the functional

$$F(x, t) = \int_{-\mathfrak{H}}^{\mathfrak{H}} \Phi(y) \mu(x, t, u, dy)$$

satisfies, for  $t \leq u$ , equation (6) and the condition  $F(x, u) = \Phi(x)$ .

Changing the sign before  $t$ , one easily obtains the formula also for the solution of an equation of type (1) in the domain  $t \geq u$ .

5°. Consider a pair of random processes  $\xi_1$  and  $\xi_2$  satisfying equations of type (5) with one and the same operator  $A$ . Under certain conditions the probability measures corresponding to these processes are mutually absolutely continuous. The formula for the density obtained in (6, 9) is also generalized (see also (3), where the measures were constructed not with the aid of stochastic equations, but with the aid of differential equations for probability distributions). For simplicity of formulation, suppose that  $\xi_1(t) = \xi_2(t)$ . Then the following result is valid.

**Theorem 5.** *Suppose that the condition is satisfied*

$$\alpha(\xi, u) = A^{-1}(\xi, u) [a_2(\xi, u) - a_1(\xi, u)] \in \mathfrak{H} \quad (\xi \in \mathfrak{H}, -t \leq u \leq \tau).$$

Then the measures  $m_1, m_2$ , corresponding to the random processes  $\xi_1, \xi_2$ , are mutually absolutely continuous and the formula is valid

$$\log \frac{dm_2}{dm_1}[\xi_1(u)] = -\frac{1}{2} \int_t^\tau \|\alpha(\xi_1(u), u)\|^2 du + \int_t^\tau (\alpha(\xi_1(u), u), dw(u)).$$

Kiev Polytechnic Institute

Received  
29 V 1965

## REFERENCES

1. Yu. M. Berezanskii, UMN, 18, 1 (109), 63 (1963).
2. G. I. Kats, Ukr. Math. J., 12, 1, 13 (1960).
3. Yu. L. Daletskii, UMN, 17, 5 (107) (1962).
4. I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Moscow, 1961.
5. E. B. Dynkin, *Foundations of the Theory of Markov Processes*, Moscow, 1959.
6. A. V. Skorokhod, *Studies in the Theory of Random Processes*, Kiev, 1961.
7. V. V. Baklan, *Equations in Variational Derivatives and Markov Processes in Hilbert Space*, Abstract of Candidate's Dissertation, Kiev, 1964.
8. T. L. Chantladze, Soobshch. AN GruzSSR, 23, 3, 529 (1964).
9. A. V. Skorokhod, *Theory of Probability and Its Applications*, 5, 45 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*