ON AN ALGORITHM FOR THE RAPID FINDING OF PSEUDOMAXIMA OF A FUNCTION DEFINED ON AN INTEGER LATTICE, WITH LIMITED MEMORY

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Abstract

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ON AN ALGORITHM FOR THE RAPID FINDING OF PSEUDOMAXIMA OF A FUNCTION DEFINED ON AN INTEGER LATTICE, WITH LIMITED MEMORY

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1. Consider the set M of nodes $X = (x_1, x_2, \dots, x_k)$ of a k-dimensional integer lattice whose coordinates satisfy the relations

$$1 \le x_k \le \dots \le x_2 \le x_1 \le n.$$

Let a function $f(X)$ be given on the set M, such that $f(Y) \ne f(X)$ if $Y \ne X$. One of the recognition algorithms proposed by L. E. Koonin leads to the problem of partitioning the set M, defined as follows. Let

$$X^I = (x_1^I, x_2^I, \dots, x_k^I)$$

be the point of the set M at which the function $f(X)$ attains its maximum. Exclude from consideration all points of the planes

$$x_i = x_j^I, \quad i, j = 1, 2, \dots, k.$$

On the remaining set, again find the point of maximum of the function $f(X)$ and, denoting it by

$$X^{II} = (x_1^{II}, x_2^{II}, \dots, x_k^{II}),$$

exclude from consideration all points of the planes

$$x_i = x_j^{II}, \quad i, j = 1, 2, \dots, k.$$

Continuing this process, we obtain a certain partition R_f of the set M. It is clear that the partition R_f is completely determined by the sequence

$$\Xi = \{X^I, X^{II}, \dots\};$$

it is also clear that the sequence Ξ consists of no more than n points.

It is understandable that, in order to determine the partition R_f, it is necessary to resort to computation of the function $f(X)$ at least Cn^k times, since, for example, all points of the set M must be examined. It is also clear that the required amount of memory cannot be less than Cn, since the sequence Ξ itself may already contain n points.

Theorem 1. There exists an algorithm for finding the partition R_f and the sequence Ξ that uses $C_1 n^k$ calls to the computation of the function $f(X)$ and requires memory of size $C_2 n$.

2. By $M(a)$ we shall denote the set of points $X \in M$ at least one of whose coordinates is equal to a.

Definition. Let N be some subset of the set M. A point $\bar{X} \in N$ is called a k-maximum on the set N if

$$f(\bar{X}) > f(Y) \quad \text{for all } Y \in N \cap \bigcup_{i=1}^{k} M(\bar{x}_i).$$

Lemma 1. Suppose that some points of the set Ξ have been found by some method and the points of the corresponding planes have been excluded from consideration. Then every k-maximum on the remaining set M' belongs to Ξ.

Indeed, a k-maximum

$$X^0 = (x_1^0, \dots, x_k^0)$$

can be excluded from the set M' only in the case where there exists such a set $M(x_s^0)$ and such a point

$$Y \in M' \cap M(x_s^0)$$

that

$$f(Y) > f(X^0).$$

The latter inequality, however, is impossible by the definition of a k-maximum.

The proposed algorithm is based on the successive finding of k-maxima.

Definition. Let N be some subset of the set M. A sequence of points

$$U^1, U^2, \dots, U^s$$

of the set N will be called **regular** if the following conditions are satisfied:

1. $f(U^1) < f(U^2) < \cdots < f(U^s)$.
2. $U^a \in \bigcup_{i=1}^{k} M(u_i^b)$; $a, b = 1, 2, \dots, k$; $a < b - 1$.

A regular sequence is called **complete** if U^s is a k-maximum on the set N.

We indicate a method for constructing complete regular sequences. Let U^1 be any point of the set N. If a regular sequence $U^1, U^2, \dots, U^i$ has already been constructed and the point U^i is a k-maximum, then a complete regular sequence has been constructed. Otherwise, as U^{i+1} we choose a point at which the maximum of the function $f(X)$ is attained on the set

$$N \cap \bigcup_{j=1}^{k} M(u_j^i).$$

It is obvious that any regular sequence is extended to a complete one in no more than n steps.

Construct on the set M a complete sequence, starting from an arbitrary point U^1: $U^1, U^2, \ldots, U^s$. By definition, U^s is a k-maximum and, by Lemma 1, $U^s \in \Xi$. Having excluded from the set M all points

$$X \in \bigcup_{i=1}^{k} M(u_i^s),$$

we obtain a certain set M^1 with the regular sequence $U^1, U^2, \ldots, U^{s-2}$ already constructed in it (it is easy to see that all these points were not among the excluded ones). If the sequence $U^1, U^2, \ldots, U^{s-2}$ is already complete, then $U^{s-2} \in \Xi$; having excluded from the set M^1 all points

$$X \in \bigcup_{i=1}^{k} M(u_i^{s-2}),$$

we obtain the set M^2 and the regular sequence $U^1, U^2, \ldots, U^{s-4}$ in it, and so on.

Suppose that in this way we have obtained some set M^l with a regular but incomplete sequence $U^1, U^2, \ldots, U^{s-2l}$ in it. Extending this sequence to a complete one, we obtain a new k-maximum, and so on. We shall call the union of all complete regular sequences constructed in this way with initial point U^1 the **tree** U.

Let the tree U contain m_u complete regular sequences and, consequently, m_u k-maxima.

Lemma 2. *The total number of vertices of the tree U does not exceed $2m_u$.*

Indeed, let U^α be an arbitrary vertex of the tree U that is not a k-maximum. Then the point U^α at some moment was excluded from consideration together with some k-maximum U^β. However, at each moment the tree contains (and as yet has not been excluded) only vertices belonging to one regular sequence. From the definition of a regular sequence it is clear that the vertex U^α can only immediately precede the k-maximum U^β. Thus all vertices of the tree (except, perhaps, the vertex U^1) are partitioned into disjoint pairs consisting of k-maxima and the points preceding them, which proves the lemma.

Corollary. *The construction of a tree U containing m_u k-maxima requires no more than $Cm_u n^{k-1}$ calls to the computation of the function $f(X)$.*

Indeed, each step in constructing the tree is connected with finding the maximum of the function $f(X)$ on some subset of one of the sets $M(a)$. At each vertex U^i of the tree U one has to bring into consideration k such sets, and moreover as many times as there exist k-maxima of the form U^{i+2}. Since in this case the number of vertices of the tree does not exceed $2m_u$, and the number of lattice points contained in the set $M(a)$ is equal to C_{n+k-2}^{k-1}, we have

$$C = \frac{3k}{(k-1)!}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{k-2}{n}\right) \leq \frac{3k}{(k-1)!}e^{2/3\,k^2/n}.$$

Suppose that the tree U turns out to be completely constructed before all points of the set M have been excluded (and, consequently, before all points—

the aggregate Ξ will have been obtained). Denote by M_u the set of excluded points. On the set $M \setminus M_u$, by means of the process described above, one can construct a new tree, starting from any point V^1, the tree V. If M_v is the set of points excluded in constructing the tree V, and $M \neq M_u \cup M_v$, then our process can be continued, starting from an arbitrary point W^1 of the set $M \setminus (M_u \cup M_v)$.

Since in all no more than n trees can be constructed, with the construction of some tree T all points of the set M will be exhausted: $M = M_u \cup M_v \cup \cdots \cup M_t$; moreover, obviously, the inequality $m_u + m_v + \cdots + m_t \leq n$ holds. Combining this inequality with the consequence of Lemma 2, we obtain the proof of the first assertion of the theorem:

$$\frac{3k}{(k-1)!}\,e^{2/3\,k^2/n}(m_u + m_v + \cdots + m_t) \leq \frac{3k}{(k-1)!}\,e^{2/3\,k^2/n}n,$$

so that

$$C_1 \leq \frac{3k}{(k-1)!}\,e^{2/3\,k^2/n}.$$

To prove the second assertion of the theorem it is enough to observe that the total number of points in all the trees $U, V, \ldots, T$ does not exceed $2(m_u + m_v + \cdots + m_t)$, which, as shown above, is not more than $2n$. Since the sequence Ξ also consists of no more than n points, the required amount of memory does not exceed $3n$; $C_2 \leq 3$.

Remark 1. The constant C_2 can in fact be lowered to the value $C_2 = 2$, since it is enough for us to remember only the points of the aggregate Ξ and that ordered sequence which we extend to a complete one; moreover, if l points of the aggregate Ξ have already been found, then the length of the sequence is not more than $2n - l$.

Remark 2. The requirement $f(X) \neq f(Y)$ for $X \neq Y$ may be dropped. In this case the partition R_f is not determined uniquely, and the problem is posed as follows: find some partition R_f of the set M, the sequence $\Xi = \{X^I, X^{\mathrm{II}}, \ldots\}$ of which satisfies the conditions $f(X^I) \geq f(X^{\mathrm{II}}) \geq \cdots$. Theorem 1 is evidently also valid for this case.

Remark 3. The problem formulated in § 1 can be generalized in the following way. Let a function $f(X)$ be given, defined on the set K of vertices $X(x_1, \ldots, x_k)$ of a k-dimensional integer lattice lying in the cube $1 \leq x_i \leq n$. Let G be some subgroup of the group P_k of permutations of k elements. Suppose further that it is known that $f(Y) = f(X)$ for all and only those points Y whose coordinates are obtained from the coordinates of the point X by permutations belonging to the subgroup G.

Denote by M the set obtained from the set K after its factorization by the subgroup G. Then the partition R_f of the set M and the sequence Ξ are defined in complete analogy with how this was done in § 1. Theorem 1 carries over entirely to this case. We note only that, instead of the sets $M(a)$, one must consider the sets obtained after factorization by the subgroup G from the sets of points of the planes $x_i = a$.

We also note that the value of the constant C_1 depends, generally speaking, on the group G; the value of C_2, however, does not exceed 2.

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Note: Figure translations are in progress. See original paper for figures.

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